Identified Parameters, Parameters of Interest and Their Relationships

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INTRODUCTION AND FUNDAMENTAL CONCEPTS

The goal of this commentary is to provide some additional results to the interesting and provocative paper of Maris and Bechger (this issue). More specifically, we have three aims. First, we want to distinguish between three fundamental concepts that are important in studying identification in statistical models: the *statistical model,* the *identified parametrization*, and the *parameters of interest*. Second, we want to take the analysis of Maris and Bechger (this issue) one step further by showing what restrictions are needed to identify the 3PL with discriminations equal to 1 (which, following San Martín, Del Pino, and De Boeck, 2006, we term 1PL-G) in a meaningful way. Third, we want to point to an implicit problem in the analysis of Maris and Bechger (this issue), but one that has much broader consequences than appear at first sight.

Let us begin with explaining the three fundamental concepts (statistical model, identified parametrization and parameters of interest) that we think are necessary to be distinguished in any identification enterprise. These concepts will be reviewed and explained using a more convenient, but otherwise equivalent, parameterization of the Rasch model as it is specified in Maris and Bechger (this issue):

$$\Pr(Y_{pi}=1)=\pi_{pi}=\frac{\exp(\theta_p)\exp(-\delta_i)}{1+\exp(\theta_p)\exp(-\delta_i)}=\frac{\in_p/\eta_i}{1+\in_p/\eta_i}, \tag{1}$$

where $\{Y_{pi}: p = 1, \ldots, P, i = 1, \ldots, I\}$ are mutually independent (i.e., within all possible combinations of different Y_{pi}'s in pairs, triplets, etc., independence holds). In this parametrization,

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both $\in_p$ and η_i are defined to be positive real numbers, and it is their ratio that determines the probability of a correct response. This parametrization is better suited for studying the identification concepts and the interpretation that follows from the identification restrictions. Note that each Y_{pi} follows (independently) a Bernoulli distribution with success probability π_{pi}.

Following Fisher (1922; see also, among others, Basu, 1975; Bamber & van Santen, 2000; Cox and Hinkley, 1974; McCullagh, 2002), a *statistical model* is a family of sampling distributions indexed by a parameter or a parameter vector. This means that when we consider a statistical model, our attention not only should be focused on a probability distribution (which is a Bernoulli distribution in the case of the Rasch model), but also on the parameters indexing it. In the case of the Rasch model, for each person p and each item i, the statistical model corresponds to a family of Bernoulli distributions indexed by $(\in_p,\eta_i)$ (and the point $[\in_p,\eta_i]$ lies in the positive quadrant of the real plane). Taking into account that the binary random variables Y_{pi} are mutually independent, the statistical model corresponding to the $P \times I$ random matrix $\boldsymbol{Y}$ is fully specified if the sampling distributions for each pair person-item are specified. Consequently, the statistical model corresponding to the full random matrix $\boldsymbol{Y}$ is the family of sampling distributions indexed by $(\in_1, \ldots, \in_P, \eta_1, \ldots, \eta_I)$, which is a point in the positive section of the real $P \times I$-dimensional space (formally denoted as $\mathbb{R}_+^n \times \mathbb{R}_+^I$). In applications of the Rasch model, one is interested in the value of the parameters $(\in_1, \ldots, \in_P, \eta_1, \ldots, \eta_I)$, as they are typically interpreted as the ability of a person and the difficulty of a item (see De Boeck & Wilson, 2004). They are, accordingly, called the *parameters of interest*. For a more detailed discussion of this concept, we refer to Engle, Hendry, and Richard (1983, section 2.3).

However, it can be seen from Equation 1 that the parameters of interest in a Rasch model are unidentified. As a matter of fact, applying the definition of parameter identification as reviewed by Maris and Bechger (this issue, Section 1), it can easily be verified that the two different parameter vectors $(\in_1, \ldots, \in_P, \eta_1, \ldots, \eta_I)$ and $(\in_1^*,\ldots,\in_P^*,\eta_1^*,\ldots,\eta_I^*)$, where $\in_p^* = c \in_p$ and $\eta_i^* = c\eta_i$ (with $c > 0$ to ensure that the numbers remain positive) result in the same probability of succes π_{pi} and hence in the same Bernoulli distribution for the random variable Y_{pi}. All of this is well known.

What is perhaps less known is that when we assume that the probabilities π_{pi} are not constrained, as is it the case in the Rasch model, our statistical model indexed by π_{pi} is identified. If we denote the realization of Y_{pi} as y_{pi} (and y_{pi} can be zero or one), then the probability $\Pr(Y_{pi} = y_{pi} \mid \pi_{pi}) = \pi_{pi}^{y_{pi}} (1-\pi_{pi})^{1-y_{pi}}$. Now, if we choose a different $\tilde{\pi}_{pi}$, the probability $\Pr(Y_{pi} = y_{pi} \mid \pi_{pi}) \neq \Pr(Y_{pi} = y_{pi} \mid \tilde{\pi}_{pi})$. Since the π_{pi}'s are identified, it should accordingly be called an *identified parameterization*. All statistical models are indexed by an identified parametrization, which does not coincide with the parametrization of interest. The latter one is a (non-injective) function of the first one, as was the case in the Rasch model.

Two fundamental questions arise from this fact: (1) under which conditions the parameters of interest are identified? (2) what is the statistical meaning of the parameters of interest after having imposed the necessary identification restrictions?

To answer the first question, we need to follow an identification strategy. A possible strategy is the one suggested by Maris and Bechger (this issue): restricting the inferences to identifiable functions of π_{pi}. A way to do so consists in introducing restrictions that lead to an injective relation from the parameters of interest to the identified parameterization. An injective relation between two sets A and B maps distinct elements from A onto distinct elements

in B. Hence, if we can find such an injective relation between the parameters of interest and the identified parameters, then no two different parameters or parameter vectors of interest lead to the same identified parametrization. Without an identification restriction, this is exactly what happens in the Rasch model. Different values for the $\in_p$'s and η_i's give rise to the same π_{pi}'s and thereby lead to an identification problem (i.e., the relation is not injective). Stated otherwise, we want to constrain the parameters of interest in such a way so that they are unique functions of the identified parameters. In the case of the Rasch model, it is therefore necessary to introduce a restriction leading to establish an injective relation between $(\in_1, \ldots, \in_P, \eta_1, \ldots, \eta_I)$ and the identified parametrization $(\pi_{11}, \ldots, \pi_{pi}, \ldots, \pi_{PI})$. Once this relation is established, it can be concluded that the parameters of interest are identified by the observations $\boldsymbol{Y}$.

From equality (1), it follows that

$$\frac{\in_p}{\eta_i} = \frac{\pi_{pi}}{1-\pi_{pi}}; \tag{2}$$

Therefore, if $\eta_1 = 1$, then $\pi_{p1}/(1 - \pi_{p1}) = \in_p$. Thus, $\in_p$ $(p = 1, \ldots, P)$ becomes identified and, from equality (2), the identification of each η_i follows.

The second question had to do with the interpretation of the parameters. The imposed identification restriction has an influence on the interpretation of the parameters of interest. Thus, $\in_p$ which is typically called the "ability of person p" represents, under the identification restriction $\eta_1 = 1$, the "betting odds" of a correct answer to the standard item 1. For instance, if for a certain person p, $\in_p = 3$, it means that we think that the odds this person solves the item are 3 to 1. Thus, if $\in_p > 1$ (resp. $\in_p < 1$), then for person p, his/her probability to correctly answer the standard item 1 is greater (resp. lesser) than his/her probability to answer it incorrectly.

Pursuing Rasch's (1966) argument, the statistical meaning of η_i can be recovered, namely for each person p

$$\eta_i = \in_p \cdot \frac{1-\pi_{pi}}{\pi_{pi}} = \frac{\pi_{p1}}{1-\pi_{p1}} \cdot \frac{1-\pi_{pi}}{\pi_{pi}} = \frac{\Pr(Y_{p1}=1)}{\Pr(Y_{p1}=0)} \cdot \frac{\Pr(Y_{pi}=0)}{\Pr(Y_{pi}=1)}, \quad i = 2, \ldots, I.$$

Thus, what is called "difficulty of item i" corresponds to an odd ratio between item 1 and item i for each person p. Hence, the difficulty parameter η_i of item i actually measures an association between item i and the standard item. If $\eta_i > 1$, then the odds of a correct answer of person p to the standard item 1 are larger than the odds of a correct answer to item i (note that the larger the η_i, the larger the difficulty). In other words, it is more probable for person p to correctly answer the standard item than the item i (that is, $\eta_i > \eta_1 = 1$).

The main conclusion of this analysis is that the precise statistical meaning of the parameters of interest in a statistical model is only revealed after identifying them. In other words, we may think whatsoever about the meaning of $\in_p$ and η_i, but the only valid statistical meaning becomes clear after identifying the statistical model in terms of the parameters of interest. In the previous analysis of the Rasch model, we have shown that the ability $\in_p$ of person p is to be interpreted as

the odds of a correct answer on a standard item and the difficulty of an item i (not the standard item) is an odds ratio (hence, a measure of association).

Three additional remarks should be added. First, if we choose item 1 to be the standard item, we need only a single person to identify the parametrization of interest. By this we mean that the model can be identified, even if $P = 1$, although for estimation purposes more than one person is needed (this is not a contradiction because identification is a necessary condition for ensuring existence of consistent and/or unbiased estimators; see Gabrielsen, 1978; and San Martín and Quintana, 2002). At this point it may sound almost trivial, but the identification analysis of the model discussed in Maris and Bechger (this issue) will show that in that situation at least two persons are needed to identify the model. Second, we have performed the identification for a fixed effects Rasch model, because we did not make any assumption about the distribution of any of the parameters. If we turn the person abilities into random effects, other issues come into play and they will be discussed below. Third, note that we could have done the identification operation also by taking an ability $\in_1$ equal to 1. Of course, this would have solved the issue equally well numerically, but the interpretation would be reversed. The first person is now a standard person and the difficulty of an item is the odds of getting an incorrect response by the standard person. All other person abilities become odds rations expressing their association with the standard person (for each item). Given that we are usually interested in measuring persons, this way of identifying the parameters of interest does not seem to be very meaningful.

THE STATISTICAL MEANING OF THE PARAMETERS OF INTEREST IN THE 1PL-G

Maris and Bechger (this issue) prove that the parameters of interest of the 3PL with all the discrimination parameters equal to σ are not identified. The fundamental questions introduced in the previous section need to be answered in this context. In this section, we answer them for the IPL-G model, which is specified as

$$\Pr(Y_{pi} = 1) = \pi_{pi} = \gamma_i + (1-\gamma_i)\frac{\exp(\theta_p - \delta_i)}{1+\exp(\theta_p - \delta_i)},$$

where the Y_{pi}'s are mutually independent. The parameters of interest are $(\theta_1,\ldots,\theta_P,\delta_1,\ldots,\delta_I,\gamma_1,\ldots,\gamma_I)$. In this model, θ_p is viewed as the ability of person p, δ_i the difficulty of item i, and γ_i the guessing parameter for item i.

For our discussion, it is more convenient to reparametrize the model as follows: $\in_p = \exp(\theta_p), \eta_i = \exp(\delta_i), \kappa_i = 1-\gamma_i$. Now, the IPL-G can be written as follows (in terms of the probability of an incorrect response):

$$\Pr(Y_{pi} = 0) = f_{pi} = 1-\pi_{pi} = \frac{\kappa_i\eta_i}{\eta_i + \in_p}. \tag{3}$$

As in the Rasch model, $\in_p$ is an ability parameter and η_i a difficulty parameter, both of them strictly positive. The parameter κ_i is a nonguessing parameter. Our reparametrization in terms of

$(\in_p, \eta_i, \kappa_i)$ is as follows, related to the reparametrization (t_p, a_i, b_i) of Maris and Bechger (this issue) (with $\sigma = 1$): $\in_p = t_p$, $\eta_i = a_i + b_i$, $\kappa_i = b_i / (a_i + b_i)$.

By the same arguments developed in the previous section, the identified parameters are the $P \times I$ f_{pi}'s. The problem consists, therefore, in writing the parameters of interest as a (injective) function of the identified parameters f_{pi}. For the moment, consider only two persons: 1 and 2. From (3), it follows that $f_{1i} / f_{2i} = (\eta_i + \in_2) / (\eta_i + \in_1)$, which in turn implies that

$$\eta_i = \frac{\in_2 \frac{f_{2i}}{f_{1i}} - \in_1}{1 - \frac{f_{2i}}{f_{1i}}} \quad i = 1, \ldots, I, \tag{4}$$

provided that $f_{1i} \neq f_{2i}$ for all $i = 1, \ldots, I$. Combining (3) and (4), it follows that

$$\kappa_i = \frac{f_{2i}(\in_2 - \in_1)}{\in_2 \frac{f_{2i}}{f_{1i}} - \in_1} \quad i = 1, \ldots, I. \tag{5}$$

Therefore, we have at this point $2I$ equations and $2I + 2$ parameters of interest (i.e., the unknowns), namely $(\in_1, \in_2, \eta_1, \ldots, \eta_I, \kappa_1, \ldots, \kappa_I)$.

Consequently, two identification restrictions should be imposed. The problem is to decide which pair of parameters of interest can be fixed. The criterium for choosing such a pair is the statistical interpretation of the remaining parameters. Let us fix η_1 and κ_1 and see what interpretation follows from it. For the time being, we do not specify values for the parameters η_1 and κ_1 but only assume that they have a predetermined value. For $i = 1$, we can now solve from Equations (4) and (5) the unknowns $\in_1$ and $\in_2$. This is possible because we are assuming that $f_{1i} \neq f_{2i}$ for all $i = 1, \ldots, I$ and thus also for $i = 1$. Therefore, $\in_1$ and $\in_2$ are identified because $\in_p = \eta_1(\kappa_1 - f_{p1}) / f_{p1}$, with $p = 1, 2$. Finally, from (3) it follows that $\in_p$ for $p = 3, \ldots, n$ are identified because it can be written as a function of f_{p1}, η_1 and κ_1 in the same way as $\in_1$ and $\in_2$:

$$\in_p = \eta_1 \cdot \frac{(\kappa_1 - f_{p1})}{f_{p1}}. \tag{6}$$

Similarly, η_i and κ_i (for $i = 1, \ldots, I$ can be solved from Equations (4) and (5) once η_1 and κ_1 are fixed.

The next step is to find what values η_1 and κ_1 should be fixed at. Analogously to the Rasch model, it makes sense to put η_1 equal to 1. Now, the parameter $\in_p$, usually called the "ability of person p," corresponds to a relative difference (in f_{p1} units) between the probability of incorrectly answering by guessing the standard item 1 and the probability that such a person incorrectly answers this item:

$$\in_p = \frac{(\kappa_1 - f_{p1})}{f_{p1}} = \frac{\pi_{p1}}{1 - \pi_{p1}} - \frac{\gamma_1}{1 - \pi_{p1}}.$$

Note that this interpretation is different from the ability in the Rasch model: it involves an additive correction due to guessing on the odds of getting the response to the standard item correctly.

Taking into account that $\in_p > 0$ for all p, Equation (6) implies that

$$\kappa_1 > \max_{1\le p\le n} \Pr(Y_{p1} = 0), \quad \text{or equivalently } \gamma_1 < \min_{1\le p\le n} \Pr(Y_{p1} = 1).$$

Once κ_1 (or, equivalently γ_1) is fixed, the probability that each person correctly answers the item 1 is above γ_1, which is the guessing probability for the standard item.

Let us now turn to what is commonly called the "difficulty parameters" (i.e., the η_i's). After imposing the restriction $\eta_1 = 1$, we may find after some algebra (using Equation (4)), that $\eta_i > \eta_1 = 1$ implies that

$$\frac{f_{2i}}{f_{1i}} > \frac{f_{21}}{f_{11}} \Leftrightarrow f_{2i}f_{11} > f_{1i}f_{21}.$$

(The relation is given here for persons 1 and 2 but it holds for any pair of persons $[p, p']$ with $p \neq p'$). Thus, item i is more difficult than the standard item 1 if the probability that both person 2 incorrectly answers the item i and person 1 incorrectly answers item 1 is greater that the probability that person 1 incorrectly answers item i and person 2 incorrectly answer item 1: it is a matter of a cross-effect between two items and two persons. Again, we can see that the interpretation of the difficulty differs from the one in the Rasch models in a counterintuitive way.

We can summarize the results from this section in the following theorem:

Theorem 1 *The parameters of interest of the 1PL-G are identified provided that (i) there exist at least two persons, such that* $f_{1i} \neq f_{2i}$ *for all* $i = 1, \ldots, I$*; (ii)* $\eta_1 = 1$*; (iii)* κ_1 *is fixed at a value greater than* $\max_{1\le p\le n} \Pr(Y_{p1} = 0)$.

The hypothesis (i) is fundamental not only for the identification analysis, but also for the interpretation of one of the parameters of interest (i.e., the difficulty η_i), which is always done referring to two persons.

IRT MODELS: FIXED-EFFECTS SPECIFICATION VERSUS RANDOM-EFFECTS SPECIFICATION

The identification analysis developed in the previous section, as well as in Section 2 of Maris and Bechger (this issue), is performed for item response theory (IRT) models under a fixed effects specification. In a fixed effects IRT model, both the person specific abilities $\{\in_p\}$ and the item specific parameters (the η_i's in the Rasch model and the η_i's and γ_i's in the IPL-G) are viewed as unknown constants. In this setup, the parameter space of the Rasch model is given by the positive quadrant of the $P + I - 1$ dimensional real space (i.e., $\mathbb{R}_+^P \times \mathbb{R}_+^{I-1}$ where the minus one occurs because η_1 is fixed). The parameter space of the IPL-G model is given by the Cartesian product of the positive quadrant of the $P + I - 1$ real space and the $I - 1$ dimensional unit hypercube (i.e., $\mathbb{R}_+^P \times \mathbb{R}_+^{I-1} \times [0,1]^{I-1}$, and the minus ones are there because both η_1 and κ_1 are fixed).

However, it is often the case that the abilities are viewed as random variables. This has several advantages: the fixed effects model is plagued by statistical difficulties (i.e., inconsistent estimators; for a recent discussion, see del Pino, San Martín, González, & De Boeck, 2008) when there are actually more persons that can be measured, and it prohibits inference to a population of persons. These problems are circumvented or solved by considering the person abilities to be a random sample from a larger population of persons. (Likewise, it sometimes makes sense to consider the items as random draws from a population of items. For more information, see De Boeck, 2008.)

If the abilities are random effects, the statistical model is obtained after integrating out the ability $\in_p$ against its population distribution. The resulting model is then called a structural or marginal model. Before marginalization, the model can be called a conditional model. The conditional model together with the random effects distribution is called a hierarchical model. When it is assumed that no parameters of the population distribution of the abilities are estimated, the parameter space of the structural Rasch model is given by $\mathbb{R}_+^{I-1}$ (if η_1 is still fixed), whereas the parameter space of the structural 1PL-G model is given by $\mathbb{R}_+^{I-1} \times (0,1)^{I-1}$ (if η_1 and κ_1 are still fixed). However, a fundamental question arises from these considerations. Is it true that the identifiability of the parameters of interest in an IRT model under a fixed-effects specification implies the identifiability of the parameters of interest in the structural IRT model?

Concering this question, we have some remarks and some additional questions. First, if the distribution of the abilities is fully known, then it is possible to prove that, for the Rasch model, the identification of the difficulty parameters in the Rasch model under a fixed-effects specification implies their identifiability in the structural Rasch model (a proof of this statement follows from San Martín & Rolin [2009], when the distribution generating the abilities is fully known). Maris and Bechger (this issue) assume in Section 3 of their paper two cases for which the random effects distribution is known. However, it is not shown that the aforementioned equivalence of identification under the fixed effects and random effects version of the model holds for the 1PL-G (i.e., the 3PL with $\sigma_i = \sigma = 1$ for all i). Second, if the distribution of the abilities is known up to a scale parameter, then there is no direct implication: it is not necessarily the case now that an identified fixed effects model leads to an identified random effects model. Here is a counterexample to show this lack of implication. Let $\boldsymbol{X}_p = (X_{p1}, X_{p2})'$, Σ be an unknown positive definite 2×2 symmetric matrix and $\alpha \neq 0$. Suppose that, for each $p = 1, \ldots, P$,

$$\boldsymbol{X}_p \overset{iid}{\sim} \mathcal{N}_2\left(\begin{pmatrix} \theta_p \\ \alpha\theta_p \end{pmatrix}, \Sigma\right). \tag{7}$$

As an illustration for this model, suppose we are measuring the left and right arm from right-armed baseball pitchers. Ideally, it is assumed that both arms have equal length (i.e., θ_p), but due to numerous pitches, the right arm has stretched somewhat and α is the stretching factor. Furthermore, suppose that the measurements are carried out by estimating the lengths from a distance and that the pitcher is in a slightly rotated position towards the observer so that the measurement errors for both judgments differ (i.e., therefore the Σ covariance matrix is left unspecified). It can be deduced that for each $p = 1, \ldots, P$, $(\alpha, \theta_p, \Sigma)$ are identified by $\boldsymbol{X}_p$.

Now, assume that our pitchers are selected randomly from a population of pitchers such that θ_p is distributed as a standard normal, namely $\theta_p \overset{iid}{\sim} \mathcal{N}(0,1)$. If we subsequently marginalize this model with respect to θ_p, we obtain:

$$\boldsymbol{X}_p \overset{iid}{\sim} \mathcal{N}_2\left(\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}\sigma_{11}+1 & \sigma_{12}+\alpha\\ \sigma_{12}+\alpha & \sigma_{22}+\alpha^2\end{pmatrix}\right), \tag{8}$$

Now it can be seen that by marginalizing we have lost the identifiability of (α, $\boldsymbol{\Sigma}$). In fact, this example resembles the urn example by Maris and Bechger (this issue), where first an urn is drawn (analoguous to our θ_p) and then a colored ball from the urn (analoguous to out X_p). In both cases, marginalization leads to unidentifiability.

Up to the best of these authors knowledge, in the context of IRT models, the implication leading to obtain the identification of the parameters of interest in a structural IRT model from the identifiability of the IRT model under a fixed effects specification has not been rigourously established although it is widely used. On the other hand, San Martín and Rolin (2009) prove the identification of the structural Rasch model (and other related models) using the statistical model that is obtained after integrating out the abilities. It remains (we believe) to prove similar results for the structural 2PL model, the structural IPL-G model, for the structural 3PL model with $\sigma_i = \sigma$, and for the structural 3PL model.

Besides the identifiability problems that may arise after marginalization over the random effects distribution, it should be noted as well that the interpretation of the parameters in the structural or marginal model changes with respect to the conditional or hierarchical model. This topic is also discussed in the mixed model literature in biostatistics. For instance, Verbeke and Molenberghs (2000, p. 24) note that "although the marginal model naturally follows from the hierarchical one, both models are not equivalent." This issue is further elaborated in the context of a random intercept random slope linear mixed model where the estimation procedure returns a zero estimate for the variance. After removing the constraint that the estimate for variances should be positive, a negative estimate for the random slope variance is obtained that is logically impossible in a random effects or hierarchical model. However, in the marginal model, the "variance of the random slope" does not have that interpretation anymore. It is just part of the structured covariance matrix for the observations, and as long as the structured covariance matrix remains positive definite, there is nothing problematic.

CONCLUDING REMARKS

Identification is typically viewed as a necessary condition for ensuring the existence of consistent and unbiased estimators. Nevertheless, identification is not only relevant at the inferential level, but also at the model specification level. At this level, its relevance deals with the actual possibility of endowing with a statistical meaning the parameters of a statistical model. This fundamental issue has been illustrated by means of two important models used in psychometrics, namely the Resch model and the 1PL-G model. Furthermore, it illustrated a fundamental difference between the identified parametrization and the parameters of interest. A statistical model always implies, by definition, an identified parametrization, but the problem is that it is not an injective transformation of the parameters of interest. The latter ones can often only be identified if additional restrictions are added.

A second aspect developed in this note is the identifiability of the parameters of interest in a 1PL-G model. Maris and Bechger (this issue) proved that the 3PL with discriminations equal to σ (in particular, equal to 1) is unidentified. Our contribution was to introduce additional restrictions for obtaining the identifiability of the parameters of interest, as well as their statistical interpretation.

The third aspect emphasized in this note is that the identification problem in an IRT model where the abilities are viewed as fixed effects is different with the structural IRT model, which is obtained after integrating out the abilities. As a matter of fact, these problems are different because the respective statistical models are different. This means that certain identification strategies used in psychometrics need to be fully qualified and, in particular, to provide proofs about the identifiability of structural IRT models.

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