Journal of Mathematical Psychology **43**, 102–122 (1999)
Article ID jmps.1998.1226, available online at http://www.idealibrary.com on IDEAL®

A Generic Disjunctive/Conjunctive Decomposition Model for *n*-ary Relations

Iwin Leenen, Iven Van Mechelen, and Paul De Boeck

University of Leuven

This paper discusses a generic decomposition model that represents an arbitrary *n*-ary relation as a disjunctive or conjunctive combination of a number of *n*-ary component relations of a prespecified type. An important subclass of order-preserving decompositions is defined and its properties are derived. The generic model is shown to subsume various known models as special cases, including the models of Boolean factor analysis, hierarchical classes analysis, and disjunctive/conjunctive nonmetric factor analysis. Moreover, it also subsumes a broad range of new models as exemplified with a novel model for multidimensional parallelogram analysis and novel three-way extensions of nonmetric factor analysis. © 1999 Academic Press

INTRODUCTION

Data in psychology often take the form of *n*-ary relations. For example, when it is observed which subjects a_1 out of a set of subjects A_1 succeed in which items a_2 out of a set of items A_2, this defines a binary relation between A_1 and A_2. As another example, when each member a_1 of a set of persons A_1 is asked to select those products a_2 he or she likes out of a set of products A_2 at different moments a_3 out of set of time moments A_3, this defines, a ternary relation between A_1, A_2, and A_3.

An important research question concerns the psychological structural mechanism or psychological process underlying such data. Such a mechanism or structure may account for (a) the *n*-ary relation between the *n* modes in itself, and (b) the natural quasi-orders on the elements of each mode as entailed by the *n*-ary relation. For example, in the case of person by item data, some ability mechanism might be involved that accounts (a) for the success/failure of any person on any item and (b) for the quasi-order on the items induced by implications between items such as "success on item *j* is a requisite for success on item *j′*" as well as for the quasi-order on the persons induced by implications between persons such as "if person *i* succeeds in an item than person *i′* also succeeds in that item." This idea has been

The authors are grateful to Jean-Paul Doignon for his helpful comments on an earlier draft of this paper. Reprint requests and correspondence concerning this paper should be addressed to Iwin Leenen, Department of Psychology, Tiensestraat 102, B-3000, Leuven, Belgium. E-mail: Iwin.Leenen@psy.kuleuven.ac.be.

0022-2496/99 $30.00
Copyright © 1999 by Academic Press
All rights of reproduction in any form reserved.

formalized in the scalogram model of Guttman (1944). In the latter model persons and items are located on a common underlying ordinal scale, with the person's position reflecting his ability and the item's position reflecting its difficulty (i.e., the minimal ability level needed to solve it). The scalogram model accounts for success/failure of any person on any item in that a person successfully solves an item if and only if (iff) the person's ability exceeds the item's difficulty; also, the model represents the quasi-order on the items in that an item j is a requisite for an item j' iff j is not more difficult than j'; similarly, the quasi-order on the persons is represented in that success of person i implies success of person i' iff i' has at least the ability level of i.

It is often unrealistic, though, to expect that a single structural mechanism or process is sufficient to account for an n-ary relation and to represent the quasi-order on each mode. Instead, several psychological structural mechanisms or psychological processes are often likely to underly the data. Person by item data, for example, will often result from the joint operation of several abilities rather than from a single one. The combined action of these underlying abilities may (a) account for the binary relation between persons and items and (b) reflect the quasi-orders on the persons and on the items. The first aspect (a) is, for example, formalized in Coombs and Kao's (1955) conjunctive model for nonmetric factor analysis (which generalizes Guttman's scalogram model). In this model, persons and items are located on each of r underlying Guttman scales, a person succeeding in an item iff his ability exceeds the item's difficulty on all scales. Regarding the second aspect (b), it may be noted that Coombs and Kao's model can be restricted to represent the quasi-orders on the persons and on the items in that, for example, item j is a requisite for item j' iff the difficulty of j does not exceed the difficulty of j' on any of the scales.

Models that imply an analysis of a data relation in terms of components (representing underlying elementary mechanisms or processes), so that the data can be reconstructed from those components, may be called *decomposition models*. Besides the models for nonmetric factor analysis of Coombs and Kao (1955), several other decomposition models have been proposed, including the hierarchical classes models (De Boeck & Rosenberg, 1988; Van Mechelen, De Boeck, & Rosenberg, 1995) and the model for matching relations (Doignon & Falmagne, 1984). Decomposition models that also represent the quasi-orders on the elements within the modes may be called *order-preserving decomposition models*. Examples of the latter again include the hierarchical classes models of De Boeck and Rosenberg (1988) and Van Mechelen *et al.* (1995).

Although the structural components considerably differ across the various decomposition models, Chubb (1986) and Koppen (1989) unified them into a common framework (for the case of binary data). In the present paper, we generalize this framework and propose a generic decomposition model for n-ary relations. Subsequently, we discuss the family of order-preserving decomposition models as a subclass within the generic model. The advantage of a generic approach as taken here is twofold: First, it allows us to identify and answer issues common to the different instantiations of the generic model, including questions on the existence of a decomposition for any data set, on the minimum number of components needed

for a decomposition (if one exists), and on methods to identify the components of a minimal decomposition. Second, the generic framework allows us to unify various existing models and may give rise to a broad range of new models.

The rest of this paper is organized as follows: Section 1 presents the generic decomposition model and the subclass of order-preserving decomposition models. Section 2 shows how three existing models are an instantiation of the generic model and illustrates how the generic model may give rise to the formulation of several new models. Section 3 contains some concluding remarks.

1. THE GENERIC DISJUNCTIVE/CONJUNCTIVE DECOMPOSITION MODEL

In this section, first the generic model is described and the duality between disjunctive and conjunctive types of the model is outlined. Second, the class of order-preserving decomposition models and some of its existence properties are discussed. Third, the dimensionality issue is dealt with.

1.1. Model

We first introduce some terminology and notation. We consider n ($n \geqslant 2$) finite sets $A_1, ..., A_q, ..., A_n$. The *Cartesian product* of these sets is denoted by $\times_{q=1}^{n} A_q$. To avoid superfluous notation, $\times_{q=1}^{n} A_q$ will be abbreviated as $\times A$. An *n-ary relation* D between $A_1, ..., A_n$ is an arbitrary subset of $\times A$. The *complement* $\bar{D}$ of an n-ary relation D equals $\times A \backslash D$.

Given n sets $A_1, ..., A_n$, one may consider a family $\mathfrak{R}$ of n-ary relations between $A_1, ..., A_n$. This means that $\mathfrak{R}$ is a subset of the power set of the Cartesian product $\times A$, or symbolically, $\mathfrak{R} \subseteq 2^{\times A}$. The generic decomposition model then may be defined as follows:

DEFINITION 1.1. Given an arbitrary set $\mathfrak{R}$ of relations between sets $A_1, ..., A_n$, a disjunctive (resp. conjunctive) $\mathfrak{R}$-decomposition model of an n-ary relation $D \subseteq \times A$, is a set of r n-ary component relations, $R_1, R_2, ..., R_r$ with $R_i \in \mathfrak{R}$ for $i = 1, ..., r$, such that $D = \bigcup_{i=1}^{r} R_i$ (resp. $D = \bigcap_{i=1}^{r} R_i$).

In the discussion of his Ph.D. thesis, Koppen (1989) described this model for the case of binary relations. Definition 1.1 is a straightforward extension of Koppen's model to n-ary relations. Also in other domains, as, for example, combinatorial graph theory (Freeman, 1983; Cozzens & Roberts, 1989), a similar decomposition idea was developed.

The following propositions, which generalize a proposition of Doignon, Ducamp, and Falmagne (1984), give a condition that is to be satisfied for $\mathfrak{R}$, in order for a disjunctive (conjunctive) $\mathfrak{R}$-decomposition to exist for any arbitrary relation D.

PROPOSITION 1.1a. *Given a family* $\mathfrak{R}$ *of relations between sets* $A_1, ..., A_n$, *any relation* D *between the same sets admits a disjunctive* $\mathfrak{R}$*-decomposition iff* $\forall (a_1, ..., a_n) \in \times A$: $\{(a_1, ..., a_n)\} \in \mathfrak{R}$.

Throughout this paper, easy proofs or straightforward extensions of earlier proofs will be omitted.

PROPOSITION 1.1b. *Given a family* $\mathfrak{R}$ *of relations between sets* $A_1, ..., A_n$, *any relation* D *between the same sets admits a conjunctive* $\mathfrak{R}$*-decomposition iff* $\forall (a_1, ..., a_n) \in \times A : \times A \backslash \{(a_1, ..., a_n)\} \in \mathfrak{R}$.

When a decomposition of a relation D is possible, the minimal number of component relations needed for a decomposition is defined as the $\mathfrak{R}$-*dimension* of that relation:

DEFINITION 1.2. The $\bigcup$-$\mathfrak{R}$-*dimension* (*resp.* $\bigcap$-$\mathfrak{R}$-*dimension*) of a relation $D \subseteq \times A$, denoted $\bigcup$-$\mathfrak{R}$-dim D (resp. $\bigcap$-$\mathfrak{R}$-dim D), is the minimum cardinality of a subset of $\mathfrak{R}$ that yields a disjunctive (resp. conjunctive) $\mathfrak{R}$-decomposition of D.

A subclass of families $\mathfrak{R}$ that may be interesting from a substantive viewpoint are those that can be defined in terms of a quantification of the elements in the n modes. For example, for the models of nonmetric factor analysis the component relations can be defined in terms of an ordinal quantification of both persons and items (see Subsection 2.2). Other types of quantification include dichotomous and nominal (multiple category) quantifications (see, for example Subsection 2.1).

With respect to the relationship between disjunctive and conjunctive decompositions, it may further be noted that in general,

$$\overline{\bigcap_{i=1}^{r} R_i} = \bigcup_{i=1}^{r} \overline{R_i}.$$

Hence, $\{R_1, R_2, ..., R_r\}$, with $R_i \in \mathfrak{R}$ $(i = 1, ..., r)$, is a disjunctive $\mathfrak{R}$-decomposition of D iff $\{\overline{R_1}, \overline{R_2}, ..., \overline{R_r}\}$ is a conjunctive $\bar{\mathfrak{R}}$-decomposition of $\bar{D}$ (with $\bar{\mathfrak{R}} = \{\bar{R} \mid R \in \mathfrak{R}\}$). So, any disjunctive $\mathfrak{R}$-decomposition of D can be transformed into a dual conjunctive $\bar{\mathfrak{R}}$-decomposition of $\bar{D}$ and vice versa. This implies that any proposition that has been proven for the disjunctive case can be dually transformed to the conjunctive case.

In general, given a relation D and a family $\mathfrak{R}$, eight models can be considered, that can be grouped into four dual pairs, as indicated in Table 1.

1.2. Order Preserving Decompositions

Any binary relation D between sets A_1 and A_2 induces natural quasi-orders or implication relations on A_1 and A_2. For example, as regards A_1, one may consider for each element a_1 of A_1, the subset of elements of A_2 that are associated with a_1

TABLE 1

Four Pairs of Dual Models given $\mathfrak{R}$ and D

Disjunctive	Conjunctive
$\mathfrak{R}$-decomposition of D	$\bar{\mathfrak{R}}$-decomposition of $\bar{D}$
$\mathfrak{R}$-decomposition of $\bar{D}$	$\bar{\mathfrak{R}}$-decomposition of D
$\bar{\mathfrak{R}}$-decomposition of D	$\mathfrak{R}$-decomposition of $\bar{D}$
$\bar{\mathfrak{R}}$-decomposition of $\bar{D}$	$\mathfrak{R}$-decomposition of D

in D. The inclusion relation between these subsets of A_2 defines a quasi-order on A_1. The quasi-orders on A_1 and A_2 often are of high substantive interest. For example, in case A_1 and A_2 refer to persons and items and D to success/failure, the quasi-order on the items corresponds to the requisite relations that play a role in knowledge space theory (see, e.g., Falmagne, Koppen, Vilano, Doignon, & Johannesen, 1990). Other examples, that illustrate the substantive interest of those quasi-orders, can be found in a paper by Gara and Rosenberg (1979) in the domain of social psychology and in Ganter and Wille (1996) in the domain of categories and concepts.

The next definitions extend the quasi-order concept to n-ary relations:

DEFINITION 1.3. Let D be a relation between n sets $A_1, ..., A_n$. For any $a_q \in A_q$ $(1 \leqslant q \leqslant n)$, the relation $D_q(a_q)$ is a $(n-1)$-ary relation, for which it holds that

$$D_q(a_q) = \{(a_1, ..., a_{q-1}, a_{q+1}, ..., a_n) \in A_1 \times \cdots \times A_{q-1} \times A_{q+1} \times \cdots \times A_n | (a_1, ..., a_{q-1}, a_q, a_{q+1}, ..., a_n) \in D\}.$$

When decomposing D, one may wish the quasi-order on A_q also to be represented in each of the component relations of the decomposition:

DEFINITION 1.4. A decomposition of a relation D between sets $A_1, ..., A_n$, into elements R_i $(i = 1, ..., r)$ of $\mathfrak{R} \subseteq 2^{\times A}$ is called *A_q-order-preserving* $(1 \leqslant q \leqslant n)$ iff for $\forall a_q, a'_q \in A_q$, it holds that

$$D_q(a_q) \subseteq D_q(a'_q) \qquad \text{iff} \quad \forall i = 1, ..., r: R_{iq}(a_q) \subseteq R_{iq}(a'_q).$$

A decomposition of a relation D between sets $A_1, ..., A_n$, into elements R_i $(i = 1, ..., r)$ of $\mathfrak{R} \subseteq 2^{\times A}$ is called *order-preserving* iff the decomposition is A_q-order-preserving for all q $(1 \leqslant q \leqslant n)$. It should be clear that the existence of a decomposition per se does not guarantee the existence of an order-preserving decomposition: For example, if $\mathfrak{R} = \{(a_1, a_2) \mid a_1 \in A_1, a_2 \in A_2\}$ (for two sets A_1 and A_2), an order-preserving decomposition of $D = A_1 \times A_2$ (as well as of most other D's) does not exist. Examples of families $\mathfrak{R}$ for which order-preserving decompositions exist for any D will be discussed in Section 2. The following proposition discusses a sufficient and necessary condition on $\mathfrak{R}$ for an order-preserving decomposition to exist for any relation D:

PROPOSITION 1.2. *Given sets $A_1, ..., A_n$, let $\mathfrak{R}$ be a set of relations between $A_1, ..., A_n$. Then an A_q-order-preserving $(1 \leqslant q \leqslant n)$ disjunctive (resp. conjunctive) $\mathfrak{R}$-decomposition exists for any relation $D \subseteq \times A$ iff $\forall a_{q'} \in A_{q'}$ $(\forall q': 1 \leqslant q' \leqslant n, q' \neq q)$, $\forall S_q \subseteq A_q$: $((a_1, ..., a_{q-1}) \times S_q \times (a_{q+1}, ..., a_n)) \in \mathfrak{R}$ (resp. $[\times A \backslash ((a_1, ..., a_{q-1}) \times S_q \times (a_{q+1}, ..., a_n))] \in \mathfrak{R}$).*

Proof. The proof is given only for the disjunctive case; it holds for the conjunctive case by duality.

$(\Rightarrow)$ Suppose $((a_1, ..., a_{q-1}) \times S_q \times (a_{q+1}, ..., a_n)) \notin \mathfrak{R}$, and, yet, there exists an A_q-order-preserving disjunctive $\mathfrak{R}$-decomposition for any D and, hence, also for $D = (a_1, ..., a_{q-1}) \times S_q \times (a_{q+1}, ..., a_n)$. For any two elements s_q, s'_q of S_q it holds

that $D_q(s_q) = D_q(s'_q)$; because the decomposition is A_q-order-preserving, it then also holds for each component R_i $(i = 1, ..., r)$ in the decomposition that $R_{iq}(s_q) = R_{iq}(s'_q)$. The last statement implies $\forall R_i$ $(i = 1, ..., r)$, $\forall s_q, s'_q \in S_q$: if $(a_1, ..., a_{q-1}, s_q, a_{q+1}, ..., a_n) \in R_i$ then also $(a_1, ..., a_{q-1}, s'_q, a_{q+1}, ..., a_n) \in R_i$, and, hence, either $R_i = \varnothing$ or $R_i = (a_1, ..., a_{q-1}) \times S_q \times (a_{q+1}, ..., a_n) = D$. Since $(a_1, ..., a_{q-1}) \times S_q \times (a_{q+1}, ..., a_n) \notin \mathfrak{R}$, we have a contradiction which proves the first part.

($\Leftarrow$) For any D an $\mathfrak{R}$-decomposition exists by equating the R_i's with singletons within D. Any $R_i = \{(a_1, ..., a_q, ..., a_n)\}$ can be extended within D by adding all elements $(a_1, ..., a_{q-1}, u, a_{q+1}, ..., a_n)$ for which $D_q(u) \supseteq D_q(a_q)$, yielding a component relation that respects the A_q-order-preserving property. ▮

Proposition 1.2 can be generalized to the general order-preserving case as follows:

PROPOSITION 1.3. *Given sets $A_1, ..., A_n$, let $\mathfrak{R}$ be a set of relations between $A_1, ..., A_n$. Then an order-preserving disjunctive (resp. conjunctive) $\mathfrak{R}$-decomposition of an arbitrary relation $D \subseteq \times A$ exists iff for any n-tuple $(S_1, ..., S_q, ..., S_n)$ with $S_q \subseteq A_q$ $(\forall q: 1 \leqslant q \leqslant n)$: $\times_{q=1}^{n} S_q \in \mathfrak{R}$ (resp. $\overline{\times_{q=1}^{n} S_q} \in \mathfrak{R}$).*

One may wonder whether decompositions exist that simultaneously are order-preserving and have minimal dimensionality. Such decompositions are called order-preserving in the minimal dimension. We first introduce a few definitions:

DEFINITION 1.5. Let D be a relation between n sets $A_1, ..., A_n$. For any $u, v \in A_q$ $(1 \leqslant q \leqslant n)$, the relation ${}_{u^\times}D_q(v)$ is the n-ary relation

$$\{(a_1, ..., a_n) \in \times A \mid a_q = u \text{ and } (a_1, ..., a_{q-1}, v, a_{q+1}, ..., a_n) \in D\}.$$

DEFINITION 1.6. Given n sets $A_1, ..., A_n$, let $\mathfrak{R}$ be a set of n-ary relations between $A_1, ..., A_n$. $\mathfrak{R}$ is called A_q-$\bigcup$-*modifiable* (resp. A_q-$\bigcap$-*modifiable*) $(1 \leqslant q \leqslant n)$ iff for each $R \in \mathfrak{R}$, it holds that $\forall u, v \in A_q$: $R \cup {}_{u^\times}R_q(v) \in \mathfrak{R}$ (resp. $R \setminus {}_{u^\times}\bar{R}_q(v) \in \mathfrak{R}$).

$\mathfrak{R}$ is called $\bigcup$-*modifiable* (resp. $\bigcap$-*modifiable*) iff $\mathfrak{R}$ is A_q-$\bigcup$-modifiable (resp. A_q-$\bigcap$-modifiable) for all q $(1 \leqslant q \leqslant n)$.

The next proposition shows that A_q-$\bigcup$-modifiability (resp. A_q-$\bigcap$-modifiability) is a sufficient condition for the existence of an A_q-order-preserving decomposition in the minimal dimension:

PROPOSITION 1.4. *Given n sets $A_1, ..., A_n$, let $\mathfrak{R}$ be a set of n-ary relations between $A_1, ..., A_n$. An A_q-order-preserving $(1 \leqslant q \leqslant n)$ disjunctive (resp. conjunctive) $\mathfrak{R}$-decomposition of any $D \subseteq \times A$ in $\bigcup$-$\mathfrak{R}$-dim D (resp. $\bigcap$-$\mathfrak{R}$-dim D) exists, if $\mathfrak{R}$ is A_q-$\bigcup$-modifiable (resp. A_q-$\bigcap$-modifiable).*

Proof. Suppose $\{R_1, ..., R_r\}$ is an $\mathfrak{R}$-decomposition of D in the minimal dimension. Each R_i $(i = 1, ..., r)$ can be made A_q-order-preserving in the following way: while there exist $u, v \in A_q$ for which $D_q(v) \subseteq D_q(u)$ and $R_{iq}(v) \not\subseteq R_{iq}(u)$, extend R_i within D to $R_i \cup {}_{u^\times}R_{iq}(v)$. The latter modified component relation is in $\mathfrak{R}$, since $\mathfrak{R}$ is A_q-$\bigcup$-modifiable. ▮

This proposition trivially implies:

COROLLARY 1.1. *Given n sets* $A_1, ..., A_n$, *let* $\mathfrak{R}$ *be a set of n-ary relations between* $A_1, ..., A_n$. *An order-preserving disjunctive* (*resp. conjunctive*) $\mathfrak{R}$*-decomposition of any D in* $\bigcup$-$\mathfrak{R}$-dim D (*resp.* $\bigcap$-$\mathfrak{R}$-dim D) *exists, if* $\mathfrak{R}$ *is* $\bigcup$*-modifiable* (*resp.* $\bigcap$*-modifiable*).

1.3. The Dimensionality Issue

The $\bigcup$-$\mathfrak{R}$-dimension (resp. $\bigcap$-$\mathfrak{R}$-dimension) of a relation can be determined using principles from hypergraph theory. This approach can be traced back to Trotter (1983), who defined a hypergraph to find the order dimension of a partial order, and to Doignon *et al.* (1984) and Koppen (1989), who applied a similar idea within the context of binary relations. We first recall some major notions (Berge, 1989).

Let $X = \{x_1, x_2, ..., x_n\}$ be a finite non-empty set. A *hypergraph* H on X consists of X and a family of subsets of X, $\{E_1, E_2, ..., E_m\}$ with $E_i \subseteq X$ $(i = 1, ..., m)$. The elements of X are called the *vertices*, and the sets $E_1, E_2, ..., E_m$ are called the *edges* of the hypergraph. A hypergraph $H = (X, \{E_1, E_2, ..., E_m\})$ is called *simple* if:

(1) $\bigcup_{i=1}^{m} E_i = X$

(2) $\forall i, j = 1, ..., m$: $E_i \subseteq E_j \Rightarrow i = j$.

Any hypergraph H can always be transformed into a simple hypergraph H° by removing those edges that are an extension of other edges and by removing those vertices that do not occur in at least one edge. The simple hypergraph H° will further be called the *bare* hypergraph of H. A set S of vertices is called *stable* in H if it does not contain an edge. A *k-coloring* of H is a partitioning of X into k stable sets, called the *color classes*. To be k-colorable, a hypergraph may not have singleton edges (i.e., edges consisting of a single vertex). The *chromatic number* of H, denoted $\chi(H)$, is the smallest integer k, for which H admits a k-coloring. Transforming a hypergraph into its bare hypergraph does not affect the chromatic number: $\chi(H) = \chi(H^\circ)$.

A hypergraph ${}_{\cup}H_{\mathfrak{R}}(D)$ (${}_{\cap}H_{\mathfrak{R}}(D)$) that will be useful to determine $\bigcup$-$\mathfrak{R}$-dim D ($\bigcap$-$\mathfrak{R}$-dim D) can be constructed from $\mathfrak{R}$ and D in the following way:

DEFINITION 1.7a. The $\bigcup$-$\mathfrak{R}$-*hypergraph* of D, denoted ${}_{\cup}H_{\mathfrak{R}}(D)$, has as vertex set the n-tuples of D and as set of edges:

$$\{A \subseteq D \mid \nexists R \in \mathfrak{R}: A \subseteq R \subseteq D\}.$$

Stated in words: The edges of ${}_{\cup}H_{\mathfrak{R}}(D)$ are the subsets of D that cannot be extended within D to an element of $\mathfrak{R}$. They constitute a generalization of the feasible sets of Koppen (1989) as defined within the context of binary relations.

Dually, the hypergraph that can be used to determine $\bigcap$-$\mathfrak{R}$-dim D can be defined as follows:

DEFINITION 1.7b. The $\bigcap$-$\mathfrak{R}$-hypergraph of D, denoted ${}_{\cap}H_{\mathfrak{R}}(D)$, has as vertex set the n-tuples of $\bar{D}$ and as set of edges:

$$\{A \subseteq \bar{D} \mid \nexists R \in \mathfrak{R}: D \subseteq R \subseteq \bar{A}\}.$$

A coloring of ${}_{\cup}H_{\mathfrak{R}}(D)$ yields a partitioning of D into as few sets as possible, under the restriction that each set must be extendable within D to an element of $\mathfrak{R}$. As indicated in the following proposition, which generalizes a proposition of Doignon *et al.* (1984) and of Koppen (1989), this coloring is directly relevant for the dimensionality issue:

PROPOSITION 1.5a. $\bigcup$-$\mathfrak{R}$-dim $D = \chi({}_{\cup}H_{\mathfrak{R}}(D))$.

The proof is a straightforward generalization of the proof in Doignon *et al.* (1984).

In addition to the computation of the $\bigcup$-$\mathfrak{R}$-dimension of D, the $\bigcup$-$\mathfrak{R}$-hypergraph of D may also be used to construct an actual disjunctive $\mathfrak{R}$-decomposition of D in the $\bigcup$-$\mathfrak{R}$-dimension. For this it suffices to extend each color class of a minimal coloring of ${}_{\cup}H_{\mathfrak{R}}(D)$ to an element of $\mathfrak{R}$. It must be noted, however, that this $\mathfrak{R}$-decomposition is not necessarily unique, because, in general, there may exist several minimal colorings and because, in general, each color class of a coloring can be extended to several elements of $\mathfrak{R}$. The uniqueness of the $\mathfrak{R}$-decomposition for given D has been investigated for specific instances of the generic decomposition model (see, e.g., Van Mechelen *et al.*, 1995). In general, nonuniqueness does not affect the applicability of $\mathfrak{R}$-decompositions per se, but is important to take into account when a formal decomposition is supplied with a substantive interpretation. If several (minimal) $\mathfrak{R}$-decompositions of a given D exist, an additional formal or substantive criterion is needed to select the decomposition of most substantive interest. (This situation is similar to that of the factor-analytical model and the rotational freedom implied by it.) One possible formal criterion to reduce the set of admissible $\mathfrak{R}$-decompositions of a given D could be the requirement of order-preservation.

Dually, Proposition 1.5 for the conjunctive case reads as follows:

PROPOSITION 1.5b. $\bigcap$-$\mathfrak{R}$-dim $D = \chi({}_{\cap}H_{\mathfrak{R}}(D))$.

Proposition 1.5 reduces the problem of finding the $\mathfrak{R}$-dimension to the problem of computing the chromatic number of a hypergraph. With respect to computational complexity, however the latter problem is known to be *NP*-hard (Garey & Johnson, 1979), which makes it unlikely that an algorithm exists that finds the chromatic number of any arbitrary hypergraph in a time that is bounded by a polynomial function of the input size.

One may wish to construct more efficient algorithms for the coloring of ${}_{\cup}H_{\mathfrak{R}}(D)$ by making use of specific properties of $\mathfrak{R}$-hypergraphs. The next proposition, however, shows that any simple hypergraph may be obtained as the bare $\bigcup$-$\mathfrak{R}$-hypergraph of a relation D; hence, there are no general properties that distinguish hypergraphs ${}_{\cup}H^{o}_{\mathfrak{R}}(D)$ from arbitrary simple hypergraphs.

PROPOSITION 1.6. *Given n sets $A_1, ..., A_n$, for any simple hypergraph $H = (X, \{E_1, E_2, ..., E_m\})$ with $X \subseteq \times A$, there exist a relation D and a family of relations $\mathfrak{R}$ such that ${}_{\cup}H^{o}_{\mathfrak{R}}(D) = H$.*

Proof. Equate D with X, and $\mathfrak{R}$ with $\{S \subseteq X \mid \nexists E_i (i = 1, ..., m)\colon E_i \subseteq S\}$. Then, by Definition 1.7a, the next equivalence holds for each $A \subseteq X$:

$$\nexists R \in \mathfrak{R}\colon A \subseteq R \subseteq X \Leftrightarrow \exists E_i\colon E_i \subseteq A.$$

Clearly, the edges of ${}_{\cup}H^{o}_{\mathfrak{R}}(D)$ are the edges of H. ∎

Proposition 1.6 implies that it is impossible to find a generic algorithm for the computation of the $\mathfrak{R}$-dimension of an n-ary relation D that is more efficient than the algorithms for coloring arbitrary hypergraphs. For specific families $\mathfrak{R}$, however, the hypergraph ${}_{\cup}H_{\mathfrak{R}}(D)$ (resp. ${}_{\cap}H_{\mathfrak{R}}(D)$) can have specific properties (for any D) that can be used to optimize the algorithm to solve the dimensionality problem.[1] As such, Chubb (1986) shows that under a complex set of conditions a binary relation D may be reduced (in polynomial time) to its so-called $\bigcup$-*core* (resp. $\bigcap$-*core*) D^* with $\bigcup$-$\mathfrak{R}$-dim D^* equal to $\bigcup$-$\mathfrak{R}$-dim D (resp. $\bigcap$-$\mathfrak{R}$-dim D^* equal to $\bigcap$-$\mathfrak{R}$-dim D). We will reformulate here the notion of $\bigcup$-core (resp. $\bigcap$-core) in hypergraph theoretic terms and generalize it to the case of n-ary relations. We will further show that for $\bigcup$-modifiable (resp. $\bigcap$-modifiable) families $\mathfrak{R}$ the associated hypergraphs may be reduced to their $\bigcup$-core (resp. $\bigcap$-core) without affecting their chromatic number.

First, we introduce a generalization of a definition from Chubb (1986):

DEFINITION 1.8. Given n sets $A_1, ..., A_n$, and $D \subseteq \times A$, then, $\alpha_q \subseteq A_q$ $(1 \leqslant q \leqslant n)$ is an A_q-$\bigcup$-*generation* set (resp. A_q-$\bigcap$-generation set) with respect to D iff for all $a \in A_q$ there exists $\beta \subseteq \alpha_q$ such that $D_q(a) = \bigcup_{b \in \beta} D_q(b)$ (resp. $\bigcap_{b \in \beta} D_q(b)$). An A_q-$\bigcup$-*generation* set (resp. A_q-$\bigcap$-generation set) α_q is *minimal* iff no proper subset of α_q is an A_q-$\bigcup$-generation set (resp. A_q-$\bigcap$-generation set).

A minimal A_q-$\bigcup$-generation (A_q-$\bigcap$-generation) set is unique upon an interchange of elements $a_q, a'_q \in A_q$ for which $D_q(a_q) = D_q(a'_q)$.

DEFINITION 1.9. Given n sets $A_1, ..., A_n$, a relation $D \subseteq \times A$, and a family $\mathfrak{R}$ of relations between the same n sets, let α_q be the minimal A_q-$\bigcup$-generation set (resp. A_q-$\bigcap$-generation set) $(1 \leqslant q \leqslant n)$. Then, the α_q-*hypergraph* α_q-${}_{\cup}H_{\mathfrak{R}}(D)$ (resp. α_q-${}_{\cap}H_{\mathfrak{R}}(D)$) is the subhypergraph obtained from ${}_{\cup}H_{\mathfrak{R}}(D)$ (resp. ${}_{\cap}H_{\mathfrak{R}}(D)$) by omitting the vertices $(a_1, ..., a_q, ..., a_n)$ for which $a_q \notin \alpha_q$ and omitting all edges that contain at least one of these vertices.

The *core-hypergraph of* ${}_{\cup}H_{\mathfrak{R}}(D)$ (resp. ${}_{\cap}H_{\mathfrak{R}}(D)$) has as vertex set and edge set the intersection across all q of the vertex sets and edge sets of the hypergraphs α_q-${}_{\cup}H_{\mathfrak{R}}(D)$ (resp. α_q-${}_{\cap}H_{\mathfrak{R}}(D)$).

PROPOSITION 1.7. *Given n sets* $A_1, ..., A_n$, *let* $\mathfrak{R}$ *be an* A_q-$\bigcup$-*modifiable* (*resp.* A_q-$\bigcap$-*modifiable*) $(1 \leqslant q \leqslant n)$ *family of relations between* $A_1, ..., A_n$ *and* α_q *the minimal* A_q-$\bigcup$-*generation set* (*resp.* A_q-$\bigcap$-*generation set*). *Then, for any* $D \subseteq \times A$, $\chi(\alpha_q\text{-}{}_{\cup}H_{\mathfrak{R}}(D)) = \chi({}_{\cup}H_{\mathfrak{R}}(D))$ [*resp.* $\chi(\alpha_q\text{-}{}_{\cap}H_{\mathfrak{R}}(D)) = \chi({}_{\cap}H_{\mathfrak{R}}(D))$].

[1] For example, if for two sets A_1 and A_2, with A_2 linearly ordered, $\mathfrak{R}$ is defined to be $\{R \subseteq A_1 \times A_2 \mid \forall u \in A_1, \forall v \prec v' \prec v'' \in A_2$: If $(u, v) \in R$ and $(u, v'') \in R$ then $(u, v') \in R\}$, then the $\bigcup$-$\mathfrak{R}$-dimension of any D $(\subseteq A_1 \times A_2)$ can be found in polynomial time. This family $\mathfrak{R}$ can be considered a variant of the family $\mathfrak{R}_p$ discussed in Subsection 2.3.

Proof. As above, only the disjunctive case is proven. Clearly, $\chi(\alpha_q\text{-}_{\cup}H_{\mathfrak{R}}(D)) \leqslant \chi(_{\cup}H_{\mathfrak{R}}(D))$. By definition of α_q, there exists for all $(a_1, ..., a_q, ..., a_n) \in D$ an element $(a_1, ..., a_{q-1}, a'_q, a_{q+1}, ..., a_n) \in D$ such that $a'_q \in \alpha_q$. Because $\mathfrak{R}$ is A_q-$\bigcup$-modifiable, it holds for any $R \in \mathfrak{R}$ that if $(a_1, ..., a_{q-1}, a'_q, a_{q+1}, ..., a_n) \in R$ then $(R \cup {}_{a_q^{\times}}R_q(a'_q)) \in \mathfrak{R}$. Hence, each vertex $(a_1, ..., a_{q-1}, a_q, a_{q+1}, ..., a_n)$ for which $a_q \notin \alpha_q$ can be placed in the same color class as $(a_1, ..., a_{q-1}, a'_q, a_{q+1}, ..., a_n)$. Consequently, $\chi(\alpha_q\text{-}_{\cup}H_{\mathfrak{R}}(D)) \geqslant \chi(_{\cup}H_{\mathfrak{R}}(D))$. ∎

This proposition immediately implies:

COROLLARY 1.2. *Given n sets $A_1, ..., A_n$, let $\mathfrak{R}$ be a $\bigcup$-modifiable (resp. $\bigcap$-modifiable) family of relations between $A_1, ..., A_n$ and α_q the minimal A_q-$\bigcup$-generation set (resp. A_q-$\bigcap$-generation set) $(1 \leqslant q \leqslant n)$. Then, for any $D \subseteq \times A$, the core-hypergraph of $_{\cup}H_{\mathfrak{R}}(D)$ (resp. $_{\cap}H_{\mathfrak{R}}(D)$) has the same chromatic number as $_{\cup}H_{\mathfrak{R}}(D)$ (resp. $_{\cap}H_{\mathfrak{R}}(D)$).*

It must be noted that, even for core-hypergraphs, the calculation of the chromatic number is usually very tedious. For "most" families $\mathfrak{R}$, only moderate-sized data sets may yield tractable hypergraphs. As an example one may consider the algorithm developed by Koppen (1987) to solve the dimensionality problem for nonmetric factor analysis. This algorithm, which reduces the relevant hypergraph even beyond its core, seems to yield results within a feasible time period for matrices of size up to about 20×20. The determination of *exact* decompositions of data relations and the solution of the associated dimensionality problem may be considered issues of predominantly theoretical relevance, though. In realistic applications, the use of *approximate* decompositions may often be preferable, not only because of reasons of computational complexity, but also because of considerations of parsimony and of a proper account for error in the data.

2. INSTANTIATIONS OF THE GENERIC MODEL

In this section we show that three existing models can be considered special cases of the generic model described in the previous section. We successively deal with (a) Boolean factor analysis together with hierarchical classes analysis and (b) nonmetric factor analysis. The discussion of existing instantiation is in no way exhaustive. Other instantiations of the generic model include the matching relations model of Doignon and Falmagne (1984). The generic model also subsumes a broad range of new models. To illustrate this, (c) a novel variant of parallelogram analysis and (d) novel three-way extensions of nonmetric factor analysis will be introduced.

2.1. Boolean Factor Analysis and Hierarchical Classes Analysis

Both Boolean factor analysis (Mickey, Mundle, & Engelman, 1983) and hierarchical classes analysis (De Boeck & Rosenberg, 1988) decompose an $m \times n$ Boolean matrix M into an $m \times r$ Boolean matrix S and an $n \times r$ Boolean matrix P, such that

$$M = S \odot P^T,$$

with P^T denoting the transpose of P and $\odot$ denoting the Boolean matrix product (Kim, 1982).

The Boolean factor analysis model is mathematically equivalent with Chubb's (1986) rectangle $\bigcup$-representation of a binary relation. Chubb defines a rectangle as any binary relation R between sets A_1 and A_2, which is such that

$$R = \mathrm{dom}(R) \times \mathrm{ran}(R),$$

with $\mathrm{dom}(R) = \{u \in A_1 \mid R_1(u) \neq \varnothing\}$ and $\mathrm{ran}(R) = \{v \in A_2 \mid R_2(v) \neq \varnothing\}$.

A rectangle has a specific structure: Rows and columns of the corresponding matrix can be permuted such that the one-entries constitute a rectangle, as illustrated in Table 2. Boolean factor analysis of a matrix M (that represents a relation D) (or a rectangle $\bigcup$-representation of D) can be considered as a disjunctive $\mathfrak{R}_c$-decomposition of D, with $\mathfrak{R}_c$ being the collection of all rectangles $R \subseteq A_1 \times A_2$. Since any $\{(u, v)\}$ with $u \in A_1$, $v \in A_2$ belongs to $\mathfrak{R}_c$, the condition in Proposition 1.1a is satisfied and, hence, each data relation can be decomposed into a finite number of rectangles.

In the formulation of the Boolean factor and hierarchical classes model it is highlighted that the rectangle component relation itself can be defined in terms of a mode-specific dichotomous quantification: The kth rectangle is the product of a subset of A_1 (namely the elements with a value of 1 in the kth column of S) and a subset of A_2 (namely the elements with a value of 1 in the kth column of P). Eventually, this quantification can be given a substantive interpretation. For instance, in a decomposition of a relation between persons and items, a substantive interpretation can be given in terms of latent strategies with S indicating which strategies are mastered by which persons and with P indicating which strategies can be used to solve succesfully which items.

The hierarchical classes model differs from the Boolean factor analysis model in that it adds to the $\mathfrak{R}_c$-decomposition the order-preserving property. Since $\mathfrak{R}_c$ is $\bigcup$-modifiable, an order-preserving $\mathfrak{R}_c$-decomposition in the minimal dimension is possible for any relation D by Corollary 1.1. (A different proof of this was given by De Boeck and Rosenberg, 1988). An order-preserving $\mathfrak{R}_c$-decomposition of D can be obtained from any $\mathfrak{R}_c$-decomposition of D in the same dimension by maximally extending each rectangle of the decomposition within D.

Chubb (1986) argued that conjunctive $\mathfrak{R}_c$-decompositions are defined only for relations $R \in \mathfrak{R}_c$, because the intersection of a number of rectangles is again a

TABLE 2
Hypothetical Example of a Rectangle Relation

	a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	a_{26}	a_{27}
a_{11}	1	1	1	0	0	0	0
a_{12}	1	1	1	0	0	0	0
a_{13}	1	1	1	0	0	0	0
a_{14}	1	1	1	0	0	0	0
a_{15}	0	0	0	0	0	0	0

rectangle. Indeed, since the condition in Proposition 1.1b does not hold for $\mathfrak{R}_c$, conjunctive $\mathfrak{R}_c$-decompositions (as well as disjunctive $\overline{\mathfrak{R}_c}$-decompositions) do not exist for arbitrary relations D. The conjunctive $\overline{\mathfrak{R}_c}$-decomposition model, which is the dual counterpart of the disjunctive $\mathfrak{R}_c$-model, does satisfy Proposition 1.1b. Order-preserving conjunctive $\overline{\mathfrak{R}_c}$-decompositions have recently been described as conjunctive models of hierarchical classes (Van Mechelen *et al.*, 1995). The latter can be related to the Galois lattice of a binary correspondence (Birkhoff, 1940; Barbut & Monjardet, 1970; Wille, 1982; Guenoche & Monjardet, 1987; Van Mechelen *et al.*, 1995; Ganter & Wille, 1996).

The dimensionality problem in Boolean factor analysis and hierarchical classes analysis comes down to finding the Schein rank of a Boolean matrix:

DEFINITION 2.1 (Kim, 1982). The *Schein rank* of a Boolean matrix M is the least number of cross-vectors whose sum is M (a cross-vector being a matrix that represents a rectangle).

The Schein rank of a matrix M equals the chromatic number of the hypergraph ${}_{\cup}H_{\mathfrak{R}_c}(D)$ with D being the relation represented by M. Reducing ${}_{\cup}H_{\mathfrak{R}_c}(D)$ to its bare hypergraph results in a graph. If $S \subseteq D$ cannot be extended within D to a rectangle (and hence forms an edge of ${}_{\cup}H_{\mathfrak{R}_c}(D)$) then there exist elements $t, u \in A_1$ and $v, w \in A_2$ such that

$$(t, v) \in S \text{ and } (u, w) \in S \qquad \text{and} \qquad ((t, w) \in \bar{D} \text{ or } (u, v) \in \bar{D}).$$

Hence, by making ${}_{\cup}H_{\mathfrak{R}_c}(D)$ bare, each edge will be reduced to a pair of its elements. Since $\mathfrak{R}_c$ is $\bigcup$-modifiable, ${}_{\cup}H_{\mathfrak{R}_c}(D)$ can be reduced to its core-(hyper)-graph without affecting its chromatic number.

2.2. Nonmetric Factor Analysis

The nonmetric factor analysis model proposed by Coombs and Kao (1955; Coombs, 1964), originates from the scalogram model developed by Guttman (1944). Guttman studied person by item success/failure data in the intelligence domain. Scalogram analysis of such data looks for a joint ordering of persons and items, which is such that an item is solved by a person correctly iff the person's position (his ability) in the ordering exceeds the item's position (its difficulty). This leads to the next definition of Guttman scalable relations in terms of mode-specific ordinal quantifications:

DEFINITION 2.2. A relation D between sets A_1 and A_2 is said to be *Guttman scalable* iff two functions $f: A_1 \to \mathbb{R}$ and $g: A_2 \to \mathbb{R}$ can be found, such that $\forall u \in A_1, v \in A_2$,

$$(u, v) \in D \qquad \text{iff} \quad f(u) > g(v).$$

A relation that is Guttman scalable is called a biorder (or a Ferrer's relation). Ducamp and Falmagne (1969) proved:

PROPOSITION 2.1. *A relation D between sets A_1 and A_2 is a biorder iff $\forall t, u \in A_1$ and $\forall v, w \in A_2$, it holds that if $(t, v) \in D$ and $(u, w) \in D$ then $(t, w) \in D$ or $(u, v) \in D$.*

As was the case with rectangles, a matrix that represents a biorder has a specific structure: It is possible to rearrange its rows and columns such that the one's constitute a (possibly irregular) triangle, as shown in Table 3.

Multidimensional generalizations of scalogram analysis, in which persons and items are ordered on each of r dimensions, have been proposed by Coombs and Kao (1955). In particular, in the conjunctive model of nonmetric factor analysis a person succeeds in an item iff the person exceeds the item on all dimensions. This means that there are scales $\mathbf{f} = (f_1, f_2, ..., f_r)$: $A_1 \to \mathbb{R}^r$ and $\mathbf{g} = (g_1, g_2, ..., g_r)$: $A_2 \to \mathbb{R}^r$, such that $\forall u \in A_1, v \in A_2$,

$$(u, v) \in D \qquad \text{iff} \quad \forall i = 1, ..., r\text{: } f_i(u) > g_i(v).$$

The least integer r for which these scales can be found has been called the bidimension of D (Doignon *et al.*, 1984).

The conjunctive model of nonmetric factor analysis obviously is a conjunctive $\mathfrak{R}_b$-decomposition of D, with $\mathfrak{R}_b$ being the set of all biorders between A_1 and A_2. The condition in Proposition 1.1b is satisfied for $\mathfrak{R}_b$, hence any relation between two sets can be decomposed into a conjunction of biorders (see also Doignon *et al.*, 1984).

Since $\mathfrak{R}_b$ is $\bigcap$-modifiable, an order-preserving conjunctive $\mathfrak{R}_b$-decomposition in the bidimension of D is possible for any relation D. An order-preserving conjunctive $\mathfrak{R}_b$-decomposition of D can be obtained from any conjunctive $\mathfrak{R}_b$-decomposition of D in the same dimension by maximally reducing each component relation.

$\mathfrak{R}_b$ also satisfies the condition of Proposition 1.1a, hence next to conjunctive $\mathfrak{R}_b$-decompositions, disjunctive $\mathfrak{R}_b$-decompositions may be considered. The latter were described by Coombs and Kao (1955; Coombs, 1964) as disjunctive models of nonmetric factor analysis. Coombs and Kao noted that the disjunctive and conjunctive models of nonmetric factor analysis are "formally isomorphic." Within the framework of the relation between conjunctive and disjunctive decompostion models, this statement may be clarified. Given a relation D and a family $\mathfrak{R}$, eight decompositions can be taken into consideration (see Table 1). In the case of $\mathfrak{R}_b$, however, $\mathfrak{R}_b$ equals $\overline{\mathfrak{R}_b}$ (because the complement of a biorder is again a biorder).

TABLE 3

Hypothetical Example of a Triangle Relation (or Biorder)

	a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	a_{26}	a_{27}
a_{11}	1	1	0	0	0	0	0
a_{12}	1	1	0	0	0	0	0
a_{13}	1	1	1	0	0	0	0
a_{14}	1	1	1	1	0	0	0
a_{15}	1	1	1	1	0	0	0
a_{16}	1	1	1	1	1	1	0

Hence, the eight possible decompositions collapse to four: The bottom half of Table 1 is equal to the top half. This means that a conjunctive $\mathfrak{R}_b$-decomposition of D is equivalent with a disjunctive $\mathfrak{R}_b$-decomposition of $\bar{D}$; hence, when making general statements without reference to a specific D, the disjunctive and conjunctive $\mathfrak{R}_b$-decompositions may be considered "formally isomorphic." "Isomorphism" of disjunctive and conjunctive $\mathfrak{R}$-decomposition models only occurs when $\mathfrak{R} = \bar{\mathfrak{R}}$.

Doignon *et al.* (1984) show that the bidimension equals the chromatic number of some hypergraph $H(D)$. It can be shown that the bare hypergraphs $H^{\circ}(D)$ and $\bigcap H^{o}_{\mathfrak{R}_b}(D)$ are equal. Since $\mathfrak{R}_b$ is $\bigcap$-modifiable, $\bigcap H_{\mathfrak{R}_b}(D)$ can be reduced to its core-hypergraph. Koppen (1987) describes an algorithm that yields a further reduction of $\bigcap H_{\mathfrak{R}_b}(D)$ beyond the core-hypergraph.

2.3. *Parallelogram Analysis*

Parallelogram analysis was developed by Coombs (1964) as a technique to model *pick any/n* data, which result from each person out of a set of persons A_1 selecting stimuli out of a set of stimuli A_2. Coombs' model assumes (a) that each individual has a hypothetical ideal stimulus, which he prefers over all other alternatives and (b) that each person prefers most, among two alternatives, the one nearest to his ideal. If those assumptions hold, a scale for persons and items exists, such that for each person the accepted items—and only those—fall within an interval around the ideal point. This implies that mode-specific quantifications $f: A_1 \to \mathbb{R}$, $f': A_1 \to \mathbb{R}$ and $g: A_2 \to \mathbb{R}$ exist such that $\forall u \in A_1$, $v \in A_2$,

$$(u, v) \in D \qquad \text{iff} \quad f(u) < g(v) < f'(u).$$

A relation satisfying the above assumptions is called a parallelogram relation. The corresponding matrix again has a specific structure: As illustrated in Table 4, rows and columns (representing the persons and the items, respectively) can be rearranged such that the one-entries form a (possibly irregular) parallelogram. In the mathematical literature, parallelogram relations are also known as matrices having the consecutive ones property for columns (Fulkerson & Gross, 1965).

As was the case for nonmetric factor analysis, it often can be expected on theoretical grounds that choice behavior results from a combination of several characteristics, rather than a single distinguishing characteristic. This suggests the

TABLE 4

Hypothetical Example of a Parallelogram Relation

	a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	a_{26}	a_{27}
a_{11}	1	0	0	0	0	0	0
a_{12}	1	1	0	0	0	0	0
a_{13}	0	1	1	0	0	0	0
a_{14}	0	1	1	1	0	0	0
a_{15}	0	0	0	1	1	0	0
a_{16}	0	0	0	0	1	1	1

need for multidimensional generalizations of the parallelogram model, in which each person and each item are represented as points in a multidimensional space.

One such generalization has been proposed by Feger (1994). His model is a compensatory model that represents persons and stimuli as points in a low-dimensional Euclidean space and that assumes that a stimulus is chosen by a person iff the corresponding stimulus point is within a hypersphere having the person's point as a centre and the person's "latitude of acceptance" as a radius (Feger, 1994).

In substantive applications, however, it often may be reasonable to expect that choices are based on a noncompensatory (in particular: conjunctive) combination of the positions on the several dimensions (see, e.g., Beach, 1990). We therefore propose here a conjunctive parallelogram model which assumes that a person accepts a stimulus iff on each relevant characteristic (i.e., on each dimension) the stimulus falls within a specified interval (the latitude of which depends on the person and the dimension under consideration). This means that quantifications $\mathbf{f} = (f_1, f_2, ..., f_r)$: $A_1 \to \mathbb{R}^r$, $\mathbf{f}' = (f'_1, f'_2, ..., f'_r)$: $A_1 \to \mathbb{R}^r$, and $\mathbf{g} = (g_1, g_2, ..., g_r)$: $A_2 \to \mathbb{R}^r$ exist, such that $\forall u \in A_1, v \in A_2$,

$$(u, v) \in D \qquad \text{iff} \quad \forall i = 1, ..., r\text{: } f_i(u) < g_i(v) < f'_i(u).$$

The conjunctive parallelogram model is a conjunctive $\mathfrak{R}_p$-decomposition model of D, with $\mathfrak{R}_p$ the set of all parallelogram relations between A_1 and A_2. For any arbitrary relation D ($\subseteq A_1 \times A_2$) a conjunctive $\mathfrak{R}_p$-decomposition exists, because the condition in Proposition 1.1b is satisfied.

Given the set $\mathfrak{R}_p$ of parallelogram relations between A_1 and A_2, still three other $\mathfrak{R}$-decomposition models could be considered: disjunctive $\mathfrak{R}_p$-decomposition, conjunctive $\overline{\mathfrak{R}_p}$-decomposition, and disjunctive $\overline{\mathfrak{R}_p}$-decomposition. Since $\overline{\mathfrak{R}_p} \neq \mathfrak{R}_p$, these four decomposition models are different from each other. We limit the discussion here to the conjunctive $\mathfrak{R}_p$-decomposition model, which seems psychologically most relevant.

Since $\mathfrak{R}_p$ is A_1-$\bigcap$-modifiable, a conjunctive $\mathfrak{R}_p$-decomposition that is order-preserving for the persons is possible in the minimal dimension. Whether the same is true for the stimuli is unclear; as appears from the following example, $\mathfrak{R}_p$ is not A_2-$\bigcap$-modifiable:

$$R = \begin{array}{c|ccccc} & a & b & c & d & e \\ u & 1 & 1 & 1 & 0 & 0 \\ v & 0 & 1 & 1 & 1 & 0 \\ w & 0 & 0 & 1 & 1 & 1 \end{array} \qquad R\backslash_{d^\times}\bar{R}_2(b) = \begin{array}{c|ccccc} & a & b & c & d & e \\ u & 1 & 1 & 1 & 0 & 0 \\ v & 0 & 1 & 1 & 1 & 0 \\ w & 0 & 0 & 1 & 0 & 1 \end{array}$$

Clearly, R is in $\mathfrak{R}_p$, whereas $R\backslash_{d^\times}\bar{R}_2(b)$ is not.

The minimal number of parallelograms needed for a conjunctive $\mathfrak{R}_p$-decomposition of given D equals $\chi({}_{\cap}H_{\mathfrak{R}_p}(D))$. A general characteristic of ${}_{\cap}H_{\mathfrak{R}_p}(D)$ is that all its edges have cardinality greater than 2: The complement of any pair of elements of $A_1 \times A_2$ is in $\mathfrak{R}_p$. Because $\mathfrak{R}_p$ is A_1-$\bigcap$-modifiable, ${}_{\cap}H_{\mathfrak{R}_p}(D)$ may be reduced to α_1-${}_{\cap}H_{\mathfrak{R}_p}(D)$.

It may be useful to note that a parallelogram relation is the intersection of two biorders (Coombs & Smith, 1973); hence, $\bigcap$-$\mathfrak{R}_b$-dim $D \leqslant 2\bigcap$-$\mathfrak{R}_p$-dim D. Given that also $\mathfrak{R}_b \subseteq \mathfrak{R}_p$, this yields lower and upper bounds for $\bigcap$-$\mathfrak{R}_p$-dim D:

$$\tfrac{1}{2}\bigcap\text{-}\mathfrak{R}_b\text{-dim } D \leqslant \bigcap\text{-}\mathfrak{R}_p\text{-dim } D \leqslant \bigcap\text{-}\mathfrak{R}_b\text{-dim } D.$$

2.4. Three-Way Generalizations of Nonmetric Factor Analysis

This section contains some three-way generalizations of nonmetric factor analysis. As a guiding data example, one might think of person $\times$ stressful situation $\times$ stress response data in a study on individual differences in Stress tolerance.

The component relations for three-way nonmetric factor analysis are generalizations of the classic two-way biorders and will be called triorders. In this section, firstly, three types of triorders will be discussed and their interrelation will be clarified: the first type of triorder is new; the second and third have been developed earlier by Collins and Cliff (1985). Secondly, multidimensional $\mathfrak{R}$-decomposition models will be proposed, and, next, the dimensionality and order-preservance issues will be examined.

Triorder definitions are generalizations of biorder definitions. The definition of the first type of triorder that will be discussed generalizes Definition 2.2:

DEFINITION 2.3. A *metric triorder* is a relation R between sets A_1, A_2, and A_3, for which there exists quantifications $f_1: A_1 \to \mathbb{R}$, $f_2: A_2 \to \mathbb{R}$, and $f_3: A_3 \to \mathbb{R}$, such that $\forall u \in A_1$, $\forall v \in A_2$, $\forall w \in A_3$,

$$(u, v, w) \in R \qquad \text{iff} \quad f_1(u) + f_2(v) + f_3(w) > 0.$$

When applied to person by response by stressful situation data, this model may be interpreted as follows: Person u will emit response v in situation w iff his stress tolerance, $-f_1(u)$, is insufficiently high to keep the stress elicited by the situation, $f_3(w)$, below the tolerance treshold for the response, $-f_2(v)$. The set of all metric triorders between three sets A_1, A_2, and A_3 is further denoted by $\mathfrak{R}_{t1}$.

A second type of triorder was described by Collins and Cliff (1985) as their Case-1 three-way generalization of the Guttman simplex.

DEFINITION 2.4. A *Collins and Cliff Case-1 triorder* is a relation R between three sets A_1, A_2, A_3, which is such that $\forall u, u' \in A_1$, $\forall v, v' \in A_2$, $\forall w, w' \in A_3$:

If $(u, v, w) \in R$ and $(u', v', w') \in R$ then (1) $(u', v, w) \in R$ or $(u, v', w') \in R$,

(2) $(u, v', w) \in R$ or $(u', v, w') \in R$, and

(3) $(u, v, w') \in R$ or $(u', v', w) \in R$.

The set of all Collins and Cliff Case-1 triorders between three sets A_1, A_2, and A_3 is denoted by $\mathfrak{R}_{t2}$.

The meaning of Collins and Cliff's Case-1 triorder and its relation with the other kinds of triorders is clarified by the following proposition, stated here without proof:

PROPOSITION 2.2. *Let R be a relation between three sets A_1, A_2, A_3. Then the following assertions are equivalent*:

(i) *R is a Collins and Cliff Case-1 triorder*,

(ii) $\forall q (1 \leqslant q \leqslant 3)$, $\forall u, u' \in A_q$: $R_q(u) \subseteq R_q(u')$ *or* $R_q(u) \supseteq R_q(u')$.

(iii) $\exists f_q: A_q \to \mathbb{R}$ $(1 \leqslant q \leqslant 3)$ *and* $\exists F: \mathbb{R}^3 \to \mathbb{R}$ *that is strictly increasing in each of its arguments, such that* $\forall u \in A_1$, $v \in A_2$, $w \in A_3$: $(u, v, w) \in R$ *iff* $F[f_1(u), f_2(v), f_3(w)] > 0$.

A third generalization of the Guttman scale was described by Collins and Cliff (1985) as their Case-2 generalization of the Guttman simplex:

DEFINITION 2.5. A *Collins and Cliff Case-2 triorder* is a relation R between three sets A_1, A_2, A_3, which is such that $\forall u, u' \in A_1$, $\forall v, v' \in A_2$, $\forall w, w' \in A_3$:

If $(u, v, w) \in R$ and $(u', v', w') \in R$ then (1) $(u', v, w) \in R$ or $(u, v', w') \in R$ and

(2) $(u, v', w) \in R$ or $(u', v, w') \in R$.

The set of all Collins and Cliff Case-2 triorders between three sets A_1, A_2, and A_3 is denoted by $\mathfrak{R}_{t3}$. Note that, unlike for metric triorders and Collins and Cliff Case-1 triorders, A_3 is not interchangeable with A_1 or A_2.

The meaning of a Collins and Cliff Case-2 triorder is clarified by the following proposition (again stated without proof):

PROPOSITION 2.3. *Let R be a relation between three sets A_1, A_2, A_3. Then the following assertions are equivalent*:

(i) *R is a Collins and Cliff Case-2 triorder*,

(ii) $\forall u, u' \in A_q$ $(q = 1, 2)$: $R_q(u) \subseteq R_q(u')$ *or* $R_q(u) \supseteq R_q(u')$,

(iii) $\exists f_q: A_q \to \mathbb{R}$ $(q = 1, 2)$ *and* $\exists F: (\mathbb{R}^2 \times A_3) \to \mathbb{R}$ *that is strictly increasing in its first two arguments, such that* $\forall u \in A_1$, $v \in A_2$, $w \in A_3$: $(u, v, w) \in R$ *iff* $F[f_1(u), f_2(v), w] > 0$.

As indicated by Collins and Cliff (1985) (and as is clear from Propositions 2.2 and 2.3), each Case-1 triorder is a Case-2 triorder. From Definition 2.3 and assertion (iii) of Proposition 2.2, it is clear that metric triorders are Collins and Cliff Case-1 triorders. The reverse, however, is not true, as appears from the relation in Table 5 which is a Collins and Cliff Case-1 triorder but not a metric triorder. Trying to find quantifications of the elements of the three modes according to

TABLE 5

A Nonmetric Triorder between the Sets $A_1=\{u_1, u_2, u_3\}$, $A_2=\{v_1, v_2, v_3, v_4\}$, and $A_3=\{w_1, w_2, w_3, w_4\}$

u_1	v_1	v_2	v_3	v_4	u_2	v_1	v_2	v_3	v_4	u_3	v_1	v_2	v_3	v_4
w_1	1	0	0	0	w_1	1	1	1	1	w_1	1	1	1	1
w_2	1	0	0	0	w_2	1	1	0	0	w_2	1	1	1	0
w_3	1	0	0	0	w_3	1	0	0	0	w_3	1	1	1	0
w_4	0	0	0	0	w_4	0	0	0	0	w_4	1	1	1	0

Definition 2.3 yields a contradiction: u_1 and u_2 require $[f_3(w_1)-f_3(w_3)]<[f_2(v_1)-f_2(v_2)]<[f_3(w_2)-f_3(w_4)]$, whereas u_2 and u_3 require $[f_3(w_1)-f_3(w_3)]>[f_2(v_3)-f_2(v_4)]>[f_3(w_2)-f_3(w_4)]$. For $\mathfrak{R}_{t1}, \mathfrak{R}_{t2}, \mathfrak{R}_{t3}$ defined on the same three sets, it holds that $\mathfrak{R}_{t1}\subseteq\mathfrak{R}_{t2}\subseteq\mathfrak{R}_{t3}$.

Since each singleton of $A_1\times A_2\times A_3$ is an element of $\mathfrak{R}_{t1}$, $\mathfrak{R}_{t2}$, and $\mathfrak{R}_{t3}$, the respective $\mathfrak{R}_{t1}$-, $\mathfrak{R}_{t2}$-and $\mathfrak{R}_{t3}$-decompositions are possible for any $D\subseteq A_1\times A_2\times A_3$. Moreover, $\mathfrak{R}_{t1}$, $\mathfrak{R}_{t2}$ as well as $\mathfrak{R}_{t3}$ are $\bigcup$-modifiable ($\bigcap$-modifiable); hence, order-preserving decompositions in the minimal dimension exist and the hypergraphs can be simplified to their core before calculating the chromatic number.

As a substantive justification for a multicomponential triorder model, assume in the person by situation by stress response data that several kinds of stress (e.g., social anxiety, fear of failure,...) rather than a single stress dimension underly the data. For one type of stress person i may be more tolerant than person j, whereas for some other type of stress person j may be more tolerant; similarly, the ordering of the situations according to the degree of stress induction may differ across different types of stress, and the same may be true for the ordering of the responses in terms of their tolerance threshold. If it is further assumed that a person reacts with a given response in a given situation iff there is at least one type of stress that elicits the response, the resulting model is a disjunctive triorder decomposition model.

3. CONCLUDING REMARKS

Section 1 proposes a generic $\mathfrak{R}$-decomposition model with $\mathfrak{R}$ being a family of relations between the n sets $A_1, ..., A_n$. Although $\mathfrak{R}$ can be any subset of the power set of the Cartesian product of the n sets, the examples discussed in Section 2 satisfy some additional conditions.

First, except for $\mathfrak{R}_{t3}$ all families $\mathfrak{R}$ under consideration can be defined in terms of mode-specific quantifications; those quantifications can be useful for a substantive interpretation of the component relations.

Second, the relations in the families $\mathfrak{R}$ discussed in Section 2 are not subject to external constraints: For all sets A_q it holds that elements of A_q are interchangeable in the sense that if a relation R belongs to $\mathfrak{R}$ any interchange of elements of A_q again yields an element of $\mathfrak{R}$. Substantive applications in which one may wish to consider $\mathfrak{R}$-decompositions with external constraints for one or more modes exist,

though. For example, with longitudinal data one may require the quantification of the elements of the time mode to respect the natural order of the time points. Such a constraint was not taken into account by Collins and Cliff (1985) (and some follow-up studies, e.g., Byrnes & Wasik, 1991; Collins, 1991) who nonetheless applied a (one-dimensional) case-2 triorder model to longitudinal child $\times$ item $\times$ age data. An alternative model could be considered that accounts for the natural order of ages. For example, one could require the child $\times$ item binary relations across ages to constitute nested biorders, in the sense that a biorder from a previous age is a subset of a biorder of a subsequent age. However, not all singleton relations satisfy this requirement, which precludes this family of relations from disjunctive $\mathfrak{R}$-decompositions for arbitrary data sets D. Alternatively, one could consider ternary relations that for a certain age interval consist of nested child $\times$ item biorders and that are empty outside this interval. Such relations may be called timebiorders. Timebiorders can be given a substantive interpretation in terms of strategies that are only applied within a specific range of ages. Such an interpretation is in line with discontinuous views on development as formalized, for example, in the (probabilistic) Saltus model of Wilson (1989). The latter was validated on several data sets, including data from a conservation task gathered by Siegler (1981). Timebiorders can be used to construct $\mathfrak{R}$-decomposition models. (Note that a disjunctive combination of timebiorders that are empty after a certain age does not exclude a gradual increase of the set of items solved by a person with increasing ages.)

We finally note that the focus of the present paper has been on the theoretical formalization of the generic decomposition model rather than on the associated data analysis. For a number of instantiations of the generic model, data-analytic procedures have already been developed whereas for several others this is not yet the case. In general, several strategies can be considered for the data analysis. First, an $\mathfrak{R}$-decomposition can be used as a deterministic model for observed data. In that case, one may either look for an exact decomposition of an observed n-ary relation D or for an approximate decomposition. Looking for an approximate decomposition comes down to approximating the observed n-ary relation D by an n-ary relation M, with the number of discrepancies between D and M (i.e., the number of elements that belong to D but not to M or vice versa) being as small as possible and with M having a low $\bigcup$-($\bigcap$-)$\mathfrak{R}$-dimension. Approximate decompositions can be useful because of reasons of parsimony or when the data can be expected to be disturbed by error. Second, an $\mathfrak{R}$-decomposition can be used as the deterministic core of a probabilistic model, which explicitly includes a random component.

Algorithms for approximate deterministic decompositions have been developed for the models of Boolean factor analysis (Mickey *et al.*, 1983) and hierarchical classes analysis (De Boeck & Rosenberg, 1988). An approach to construct approximate biorder representations has been proposed by Koppen (1989). A probabilistic version of the hierarchical classes models as well as an associated algorithm has been proposed by Maris, De Boeck, and Van Mechelen (1996). Probabilistic extensions of conjunctive/disjunctive nonmetric factor analyis as well as associated algorithms have been proposed by Maris (1995) and by van Leeuwe and Roskam (1991).

REFERENCES

Barbut, M., & Monjardet, B. (1970). *Ordre et classification*: *Algèbre et combinatoire* (Vols. 1–2). Paris: Hachette.

Beach, L. R. (1990). *Image theory*: *Decision making in personal and organizational contexts*. Chichester: Wiley.

Berge, C. (1989). *Hypergraphs*: *Combinatorics of finite sets*. Amsterdam: North-Holland.

Birkhoff, G. (1940). *Lattice theory*. Providence: Amer. Math. Soc.

Byrnes, J. P., & Wasik, B. A. (1991). Role of conceptual knowledge in mathematical procedural learning. *Developmental Psychology*, **27**, 777–786.

Chubb, C. (1986). Collapsing binary data for algebraic multidimensional representation. *Journal of Mathematical Psychology*, **30**, 161–187.

Collins, L. M. (1991). The measurement of dynamic latent variables in longitudinal aging research: Quantifying adult development. *Experimental Aging Research*, **17**, 13–20.

Collins, L. M., & Cliff, N. (1985). Axiomatic foundations of a three-set Guttman simplex model with applicability to longitudinal data. *Psychometrika*, **50**, 147–158.

Coombs, C. H. (1964). *A theory of data*. New York: Wiley.

Coombs, C. H., & Kao, R. C. (1955). Nonmetric factor analysis. (Engineering Research Bulletin No. 38). Ann Arbor: Univ. of Michigan Press.

Coombs, C. H., & Smith, J. E. K. (1973). On the detection of structure in attitudes and developmental processes. *Psychological Review*, **80**, 337–351.

Cozzens, M. B., & Roberts, F. S. (1989). On dimensional properties of graphs. *Graphs and Combinatorics*, **5**, 29–46.

De Boeck, P., & Rosenberg, S. (1988). Hierarchical classes: Model and data analysis. *Psychometrika*, **53**, 361–381.

Doignon, J. P., Ducamp, A., & Falmagne, J. C. (1984). On realizable biorders and the biorder dimension of a relation. *Journal of Mathematical Psychology*, **28**, 73–109.

Doignon, J. P., & Falmagne, J. C. (1984). Matching relations and the dimensional structure of social choices. *Mathematical Social Sciences*, **7**, 211–229.

Ducamp, A., & Falmagne, J. C. (1969). Composite measurement. *Journal of Mathematical Psychology*, **6**, 359–390.

Falmagne, J. C., Koppen, M., Vilano, M., Doignon, J. P., & Johannesen, L. (1990). Introduction to knowledge spaces: How to build, test and search them. *Psychological Review*, **97**, 201–224.

Feger, H. (1994). *Structure analysis of co-occurence data*. Aachen: Shaker.

Freeman, L. C. (1983). Spheres, cubes and boxes: Graph dimensionality and network structure. *Social Networks*, **5**, 139–156.

Fulkerson, D. R., & Gross, O. A. (1965). Incidence matrices and interval graphs. *Pacific Journal of Mathematics*, **15**, 835–855.

Ganter, B., & Wille, R. (1996). *Formale Begriffsanalyse*: *Mathematische Grundlagen*. Berlin: Springer-Verlag.

Gara, M. A., & Rosenberg, S. (1979). The identification of persons as supersets and subsets in free-response personality descriptions. *Journal of Personality and Social Psychology*, **37**, 2161–2170.

Garey, M. R., & Johnson, D. S. (1979). *Computers and intractability*: *A guide to the theory of NP-completeness*. San Francisco: Freeman.

Guénoche, A., & Monjardet, B. (1987). Méthodes ordinales et combinatoires en analyse des données. *Mathématiques*, *Informatiques et Sciences Humaines*, **100**, 5–47.

Guttman, L. (1944). A basis for scaling qualitative data. *American Sociological Review*, **9**, 139–150.

Kim, K. H. (1982). *Boolean matrix theory and applications*. New York: Dekker.

Koppen, M. G. M. (1987). On finding the bidimension of a relation. *Journal of Mathematical Psychology*, **31**, 155–178.

Koppen, M. G. M. (1989). *Ordinal data analysis*: *Biorder representation and knowledge spaces*. Ph.D. thesis. University of Nijmegen, The Netherlands.

Maris, E. (1995). Psychometric latent response models. *Psychometrika*, **60**, 523–547.

Maris, E., De Boeck, P., & Van Mechelen, I. (1996). Probability matrix decomposition models. *Psychometrika*, **61**, 7–29.

Mickey, M. R., Mundle, P., & Engelman, L. (1983). Boolean factor analysis. In W. J. Dixon (Ed.), *BMDP statistical software* (pp. 538–545, 692). Berkeley, CA: California Press.

Siegler, R. S. (1981). Development sequences within and between concepts. *Monograph of the Society for Research in Child Development*, **46** (2, Serial No. 189).

Trotter, W. T., Jr. (1983). Graphs and partially ordered sets. In L. W. Beineke & R. J. Wilson (Eds.), *Selected topics in graph theory*, *II* (pp. 237–268). New York: Academic Press.

van Leeuwe, J. F. J., & Roskam, E. E. (1991). The conjunctive item response model. A probabilistic extension of the Coombs and Kao model. *Methodika*, **5**, 14–32.

Van Mechelen, I., De Boeck, P., & Rosenberg, S. (1995). The conjunctive model of hierarchical classes. *Psychometrika*, **60**, 505–521.

Wille, R. (1982). Restructuring lattice theory: An approach based on hierarchies of concepts. In O. Rival (Ed.), *Ordered sets* (pp. 445–470). Boston: Reidel.

Wilson, M. (1989). Saltus: A psychometric model of discontinuity in cognitive development. *Psychological Bulletin*, **105**, 276–289.

Received September 11, 1996