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Unraveling the small sample bias in AR(1) models: the pros and cons of available bias correction methods

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1 Introduction

A key goal in psychological research is to investigate the dynamics of within-person psychological processes. A popular model to unravel these dynamics is the first-order autoregressive [AR(1)] model. This model depicts temporal associations in a variable, which can be modeled by allowing a variable y_t at time t to display dependence on itself at time $t-1$. This effect is referred to as the autoregressive effect or AR effect. In psychological research, such dependence has been tied to relevant psychological concepts. For instance, in the study of affect dynamics, this dependence is interpreted as emotional inertia and regulatory behavior in affective processes (Kuppens et al., 2010). Moreover, empirical results have shown that individual differences in inertia were associated with psychological maladjustment (Koval et al., 2013) and anticipation of social stress (Koval and Kuppens, 2012). In addition, such temporal associations (i.e., autocorrelation) are prevalent in single-case experimental designs [SCEDs], which use repeated measure outcomes in a single case (Shadish and Sullivan, 2011). Here, autoregressive models are also commonly specified to handle temporal associations (Ariens et al., 2020).

Autoregressive dynamics are thus of conceptual importance in psychological research, and as such, autoregressive parameters often feature as targets for statistical inference. Basic assumptions for the autoregressive process, such as stationarity and known lag orders, are crucial for model estimation and inference. Despite the prevalent assumption of stationarity in psychological research (Ariens et al., 2020), violations of this assumption lead to biased estimates and sampling distributions, invalidating inference (Ryan et al., 2025). In addition, the assumption of a known first-order autoregressive process may not be realistic, prompting discussions of fitting models of arbitrary order to an autoregressive process of unknown order (Jacobson et al., 2019; Jebb et al., 2015). Moreover, as psychologists increasingly adopt sophisticated statistical approaches (Young, 2018), time series models that incorporate lagged associations among variables, such as the multilevel AR and vector autoregressive (VAR) models, have gained popularity (Ariens et al., 2020; Ernst et al., 2024a). Nevertheless, estimating and performing inference on these models can be problematic, since small sample biases are also found in these cases. For example, Krone et al. (2016) compared different estimation methods for the AR effect in the multilevel AR(1) model and found out that the

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bias in the estimators of the AR effects depends on the number of time points per person (i.e., T) in addition to the number of individuals.

Even in an ideal scenario where conditions like stationarity and a known true model order are satisfied, estimating the autoregressive effect in an AR(1) model for the time series of a single individual can be challenging, especially when T is small. This challenge primarily stems from small-sample bias in estimating the AR effect. It is known that the AR effect estimator is biased in small samples when using state-of-the-art estimation methods such as ordinary least squares (OLS) (Marriott and Pope, 1954), maximum likelihood estimation (Reinsel and Cheang, 2003), and Bayesian estimation with weakly informative priors (Krone et al., 2017; Patterson, 2000). For positive AR effects, which are often expected in the literature on affect dynamics (Kuppens et al., 2010), this bias will be downwards (Nickell, 1981).

In this paper, we aim to highlight the challenges that arise when working with small samples to estimate and conduct inference on the AR effect, with a focus on OLS. Haqiqatkhah et al. (2024) stated that the single-level AR(1) model is a building block of the multilevel AR(1) model. Thus, we focus on the stationary AR(1) model for a single individual, as it represents the simplest case that can be extended to more complex scenarios. We argue that the small sample problems are essential for psychological scientists to consider, since they are often faced with numerous constraints on sample size when designing their studies. For example, in experience sampling research, T is usually kept small to minimize participants' burden (Wrzus and Neubauer, 2023). For instance, while analyzing ESM data, Li et al. (2024) calculated daily negative affect (NA) inertia estimates by fitting a separate AR model to the data per day, so using 10 scores at max, and included these estimates in a later modeling step. Another field where the number of time points is usually limited is SCED. Shadish and Sullivan (2011) investigated 809 SCEDs and reported that the number of time points in these 809 SCEDs has a median of 20 with a range $[2, 160]$, and the majority of SCEDs had less than 50 data points. These examples indicate the prevalence of small samples in psychological applications. Due to the downward bias for positive AR effects, researchers may be systematically underestimating AR effects in their data by failing to take into account these small sample challenges. Importantly, biased point estimates will affect statistical inference, including hypothesis tests of the significance of the AR parameter. Yet little attention has been paid to the extent of the issue in psychological applications, nor to the methods available for addressing small-sample biases in the psychological literature.

To shed light on the impact of a small sample in the estimation of the AR effect and its implications for hypothesis testing, we focus on three main objectives: first, we introduce the AR(1) model. We describe the causes and consequences of small sample bias in estimating the AR effect. Moreover, we demonstrate the implications of the biased estimates for the sampling distribution of this estimate. Second, we discuss the advantages and disadvantages of the various bias-correction methods proposed in the time series literature. We mainly present two correction methods: Kendall's correction (Kendall, 1954) and a Bayesian estimation-based correction method (Jarocinski and Marcet, 2010). Finally, through simulation studies, we showcase the aforementioned consequences of the small sample bias. We compare the performance of the OLS estimator with these correction methods by examining various statistical outcomes, including bias, Type I error rate, and statistical power. We conclude the paper with a general discussion.

2 The AR(1) model

For a time series of $\{y_1, y_2, \cdots, y_T\}$ of a single individual, the AR(1) model is defined as:

$$y_t = \alpha + \rho y_{t-1} + \epsilon_t, \epsilon_t \overset{\text{i.i.d}}{\sim} N(0, \sigma_\epsilon^2), \tag{1}$$

where α denotes the intercept, ρ denotes the AR effect, and ϵ_t denotes the innovation or stochastic term, which is assumed to be independent and identically normally distributed

with mean zero and variance σ_ϵ^2. The AR effect ρ captures the linear relationship between the dependent variable and itself at the previous time point.

Stationarity for the stationary AR(1) model means that the mean, variance, and autocorrelation of the process remain constant over time. Stationarity represents changes over time that tend to be stable around the mean, along with some random fluctuations (i.e., an event would introduce fluctuations or sudden changes in the process. The effect of the event will decay over time as the process returns to its mean level). For the common assumption in psychology that the AR(1) process is stationary, the AR effect should be smaller than 1 in absolute value (i.e., $|\rho| < 1$). Furthermore, stationarity is a crucial assumption for valid inference (Ryan et al., 2025). Under the stationarity assumption, the marginal distribution of y_t follows a normal distribution: $N(\frac{\alpha}{1-\rho}, \frac{\sigma_\epsilon^2}{1-\rho^2})$ (Hamilton, 1994). Thus, the expected value of y_t is given by $\mu_{y_t} = \frac{\alpha}{1-\rho}$ and the variance of y_t is $\sigma_y^2 = \frac{\sigma_\epsilon^2}{1-\rho^2}$ for all $t \in \{1, \ldots, T\}$. Hence, the stationary AR(1) model implies a time-invariant mean, variance, and autocorrelation, making the model's characteristics stable over time (Andrews, 1993; Ariens et al., 2020; Lütkepohl, 2007).

State-of-the-art estimation methods for the AR effect can be categorized into two main approaches: frequentist and Bayesian. Standard frequentist methods include OLS, maximum likelihood estimation (i.e., exact MLE) (Barlow et al., 2009), Yule-Walker equations (de Hoon et al., 1996; Ernst and Shaman, 2019), and Burg's method (Stine and Shaman, 1989). For Bayesian methods, researchers must specify a prior distribution on the model's parameters. For example, methods developed in the Mplus software (Asparouhov et al., 2018; Hallquist and Wiley, 2018), a popular closed-source software widely used in psychological research, use a flat prior as the default option for estimating the AR effect (Asparouhov et al., 2018). Flat priors are one of the classical uninformative priors used in Bayesian inference for linear regression models with i.i.d. errors (Asparouhov et al., 2018). However, they are no longer considered uninformative in time series models (Phillips, 1991). In fact, it biases the estimates by treating ρ as a regression weight in a simple linear regression, yielding results similar to those of OLS (Jarocinski and Marcet, 2010). Specifically, the value of ρ influences the extent of correlation between y_t and y_{t-1}. Thus, when ρ is small, the process could return to its mean level fast when an event happens. Therefore, the amount of information contained in the sample depends on the value of ρ. However, this dependency is ignored when applying flat priors (Phillips, 1991). Moreover, specifying priors for AR models is not trivial. For example, Jarociński and Marcet (2019) compared three standard informative priors- the Minnesota prior (Litterman, 1986), the prior proposed by Sims and Zha (1998), and the Dynare prior (Adjemian et al., 2024)— and showed that these priors yield different estimates. Under these circumstances, it is hard to choose one prior over others as the interpretation of the prior is not straightforward. Specifically, commonly used priors often fall short because it is unclear how to translate parameter information into the time-series values of the AR process. For instance, for the symmetrized reference prior, the probability of obtaining higher values of $\mid \rho \mid$ for the posterior is higher compared to the flat prior or Jeffrey's prior (Berger and Yang, 1994; Krone et al., 2017; Phillips, 1991). However, we observe that the estimated AR effect is always negatively biased when $\rho = 0$ (see Figure 1), meaning that the symmetrized reference prior leads to more biased results as it tends to give higher values of $\mid \rho \mid$.

Figure 1 and 2 demonstrate that estimation performances (i.e., mean and standard deviation of the estimated AR effect) of the OLS, exact MLE, Yule-Walker equations, Burg's method, and Mplus (i.e., Bayesian with a flat prior using the Mplus software (Hallquist and Wiley, 2018)) are similar. In particular, we observed that the OLS method and Mplus overlap. For small sample sizes (i.e., $T = 5, 10, 20, 50$), all methods lead to biased estimates of the AR effect. Given that all methods exhibit similar performances and that OLS offers a straightforward way to correct for the small sample bias based on the analytical approximation of the asymptotic bias, we henceforth focus on estimating the AR effect with

OLS.

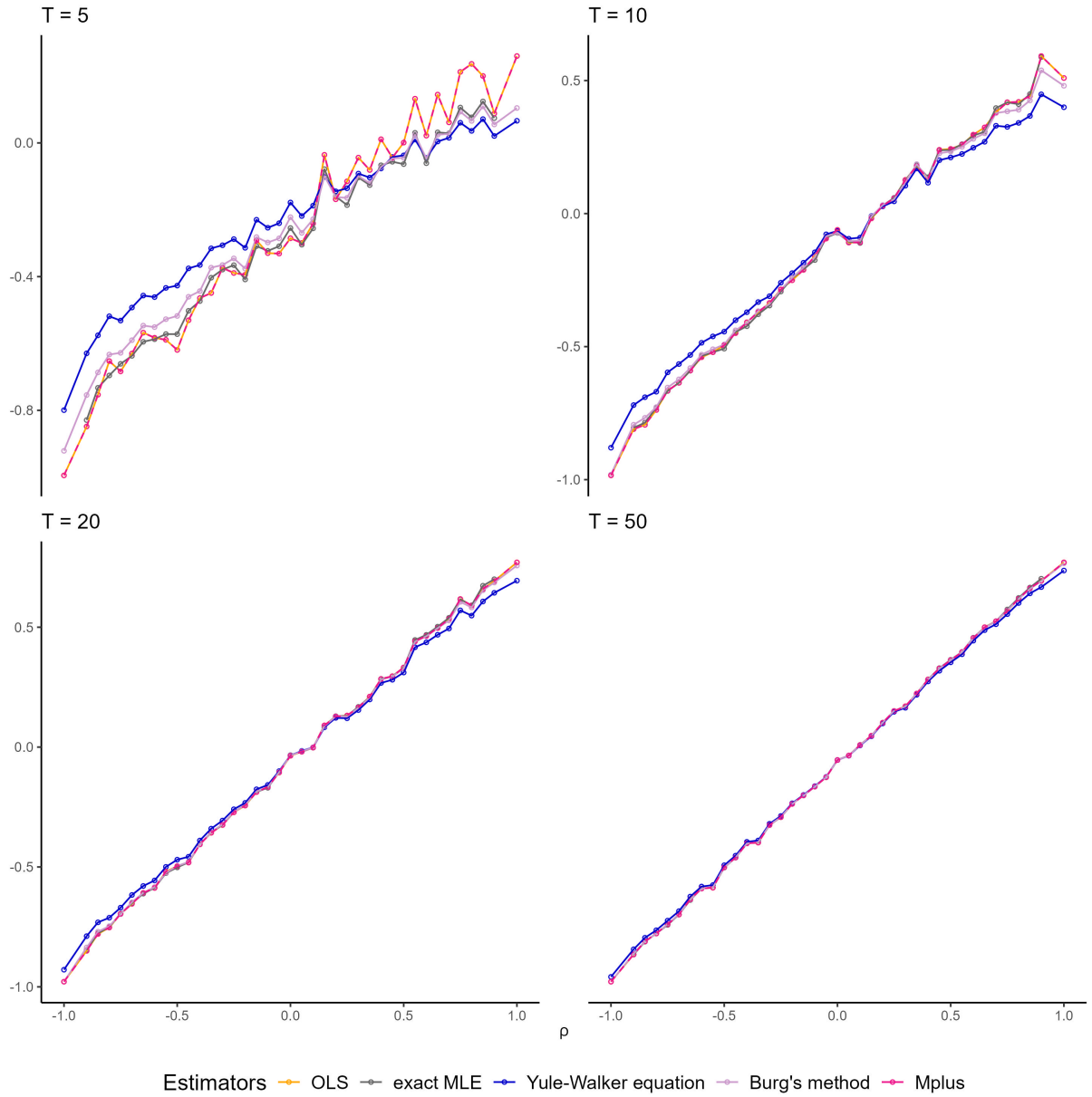


Figure 1: Comparisons between the OLS, exact MLE, Yule-Walker equation, Burg's method and Mplus (Bayesian with flat prior) for AR(1) model with $\rho \in [-1, 1]$, $\alpha = 0$, $\sigma_\epsilon = 1$: mean of the estimated AR effect across 1000 replicates with $T = 5, 10, 20, 50$.

In simple linear regression, the OLS estimator of the regression coefficient is given by

$$\hat{\rho}_{OLS} = \frac{S_{x,y}}{S_x^2}, \tag{2}$$

where $S(x, y)$ is the sample covariance between the covariate and the criterion variable. For the AR(1) model, the covariate is the lagged predictor y_{t-1}. The term $S(y_t, y_{t-1})$ is called the sample autocovariance, and can be calculated by computing the covariance between y_t and y_{t-1}. Then, for an observed time series $y_1, y_2, \ldots, y_T$, the OLS estimator of the AR effect is derived by applying the sample variance and sample autocovariance (see, e.g., (Hamilton, 1994)), similarly as in the simple linear regression:

$$\hat{\rho}_{OLS} = \frac{S_{y_t, y_{t-1}}}{S_{y_{t-1}}^2}. \tag{3}$$

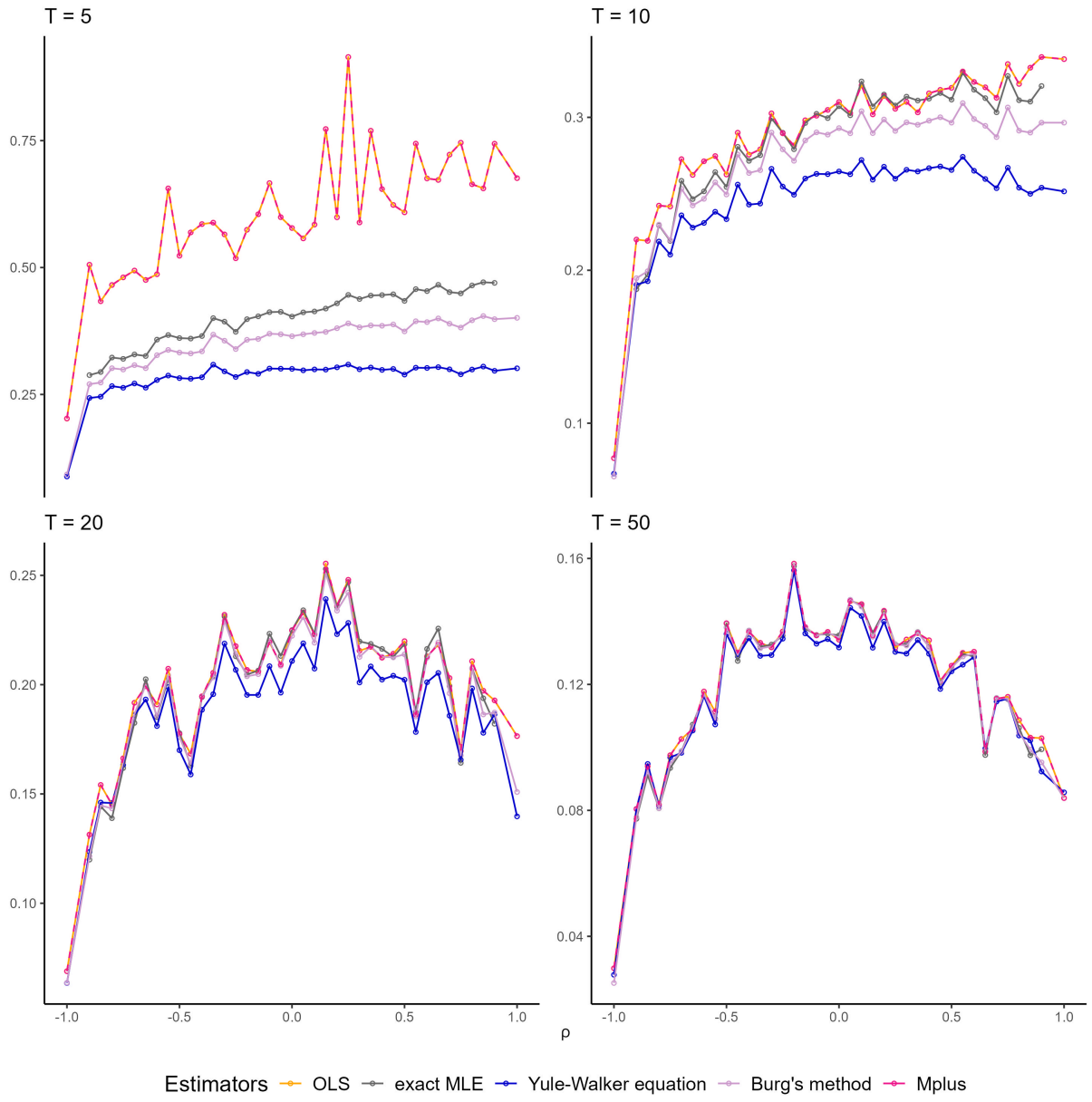


Figure 2: Comparisons between the OLS, exact MLE, Yule-Walker equation, Burg's method and Mplus (Bayesian with flat prior) for AR(1) model with $\rho \in [-1, 1]$, $\alpha = 0$, $\sigma_\epsilon = 1$: standard deviation of the estimated AR effect across 1000 replicates with $T = 5, 10, 20, 50$.

where $S^2_{y_{t-1}}$ represents the sample variance of y_{t-1} and $S_{y_t, y_{t-1}}$ represents the sample autocovariance. Meanwhile, the OLS estimator of the intercept can be computed using the formula:

$$\hat{\alpha}_{OLS} = \frac{\sum_{t=2}^{T} y_t - \hat{\rho}_{OLS} \sum_{t=2}^{T} y_{t-1}}{T - 1}. \tag{4}$$

Assume the target of a study is to test for the AR effect, so the hypothesis of testing is $H_0 : \rho = \rho_0$. The common test statistic used to test the hypothesis is the t-statistic (Student, 1908), which is

$$t = \frac{\hat{\rho}_{OLS} - \rho_0}{SE(\hat{\rho}_{OLS})}. \tag{5}$$

To obtain the test statistic, we require not only the OLS estimator but also the standard error of the OLS estimator. Typically, the default standard error of $\hat{\rho}_{OLS}$ is given by (Revol

et al., 2023):

$$SE(\hat{\rho}_{OLS}) = \sqrt{\frac{\hat{\sigma}_\epsilon^2}{(T-1)S_{y_{t-1}}^2}}. \tag{6}$$

where $\hat{\sigma}_\epsilon^2$ is the estimated variance of the residuals in the OLS regression.

3 Small sample bias of the OLS estimator of the AR effect

In this section, we discuss the small sample behavior of the OLS estimator of the AR effect in terms of bias and variance. Additionally, we investigate how the bias in the OLS estimator of the AR effect affects the estimation of the intercept α.

3.1 Bias in the OLS estimator of the AR effect

Unlike common linear regression models, a special property of the autoregressive model is the dependency between ϵ_t and current and future values of the process (i.e., $cov(\epsilon_t, y_{t+s}) = \rho^s \sigma_\epsilon^2$, $s \in \mathbb{N}, s \leq T - t$, see equation A1 in Appendix A.1). This dependence leads to issues in the estimation of the AR effect when using state-of-the-art estimation methods (Mikusheva and Sølvsten, 2024). This situation is known as a violation of the strict exogeneity assumption, which states that $E(\epsilon_t \mid y_1, \ldots, y_T) = 0$. This assumption is crucial in linear regression models, and ensures that the OLS estimator is unbiased. However, for the AR(1) model, the violation of this assumption leads to biased estimates of the AR effect. Consequently, the expected value of the estimator $E(\hat{\rho})$ differs from the true population value ρ.

Importantly, small sample biases will also be present for the methods we mentioned above (see Figure 1 and the comparison done by Krone et al. (2017)). Therefore, in this paper, we focus on the consequences of the small sample bias when estimating the AR effect using OLS. We will do this by first providing a didactical illustration. Figure 3 presents a comparison between comparatively shorter time series and a long time series, displaying an AR(1) process of 50 time points with ρ equal to 0.8 and α equal to 0. Specifically, the process was divided into five short time series, each consisting of 10 consecutive time points.

In accordance with the stationary property assumed in the previous section, the time series in Panel A, Figure 3 should fluctuate around a mean (i.e., 0 in this case). We indeed observe this phenomenon for the time series of 50 time points. Nevertheless, the fact that the data are generated under a stationary AR(1) model is less obvious for the short time series. In Panel B in Figure 3, we observe that data points of the complete time series are located along the true regression line. However, for the short time series, we observed that the data points are clustered together around the initial value [1]. Furthermore, as the sample covariance measures the joint variability of y_t and y_{t-1} (Rice, 2006) and it is the numerator in equation 3, the clustering of the observe data points around the initial value when the time series is short, leads to an underestimation of $S_{y_t, y_{t-1}}$ and $S_{y_t}^2$ (Figure 3), which is associated with an underestimation of ρ when $\rho \in (-0.3, 1)$, and an overestimation of ρ when $\rho \in (-1, -0.3)$(Arnau and Bono, 2001).

Following the definition of the AR(1) model in equation 1, we know that y_2 is correlated with y_1 plus some noise. This means that y_2 often locates somewhere close to y_1. Likewise, y_t often locates close to y_{t-1}. In the case of small T, the contribution of the cumulated innovation term in determining y_t is sometimes not sufficient for y_t to depart from the initial observation y_1. Thus, in the small sample setting, data points are clustered around the initial value. However, the contribution of the initial value y_1 eventually vanishes as the process is observed for a longer period.

The causes of the small-sample bias can be stated more formally using the definition of the OLS estimator (see equation 3). We will showcase the case in which $\alpha = 0$. In this case,

[1] For a comparison with simple linear regression, see figure 14 in Appendix B.3

the OLS estimator of ρ can be simplified from equation 3 to:

$$\hat{\rho}_{OLS} = \rho + \frac{\Sigma_2^T y_{t-1}\epsilon_t}{\Sigma_2^T y_{t-1}^2}. \tag{7}$$

As we mentioned above, the violation of the strict exogeneity assumption means that the errors are correlated with the lagged values of y_t. Thus, ϵ_t is correlated with $\sum_{\tau=2}^{T} y_{\tau-1}^2$. A precise calculation of the expected value of the OLS estimator of the AR effect is not available. However, using the Taylor expansion, we can obtain an approximation of equation 7 (see equation A13 in Appendix A.1),

$$E(\hat{\rho}_{OLS}) \approx \rho - \frac{2(1-\rho^2)\rho^{2T}\sum_{k=1}^{T-1} k\rho^{-2k-1}}{(T-1)^2} \tag{8}$$

From this equation, we observe that under the stationary assumption, the OLS estimator is consistent, as it converges to the true AR parameter in probability (Hamilton, 1994) when T is large (see equation A15 in Appendix A.1):

$$E(\hat{\rho}_{OLS}) \approx \rho \quad as \quad T \to \infty. \tag{9}$$

Furthermore, from equation 8, we can observe that the bias of the OLS estimator $\hat{\rho}_{OLS}$ depends on the value of the AR effect and the sample size. This bias is significant when T is small, but vanishes when T is large. Specifically, equation 8 could be reformulated as (see

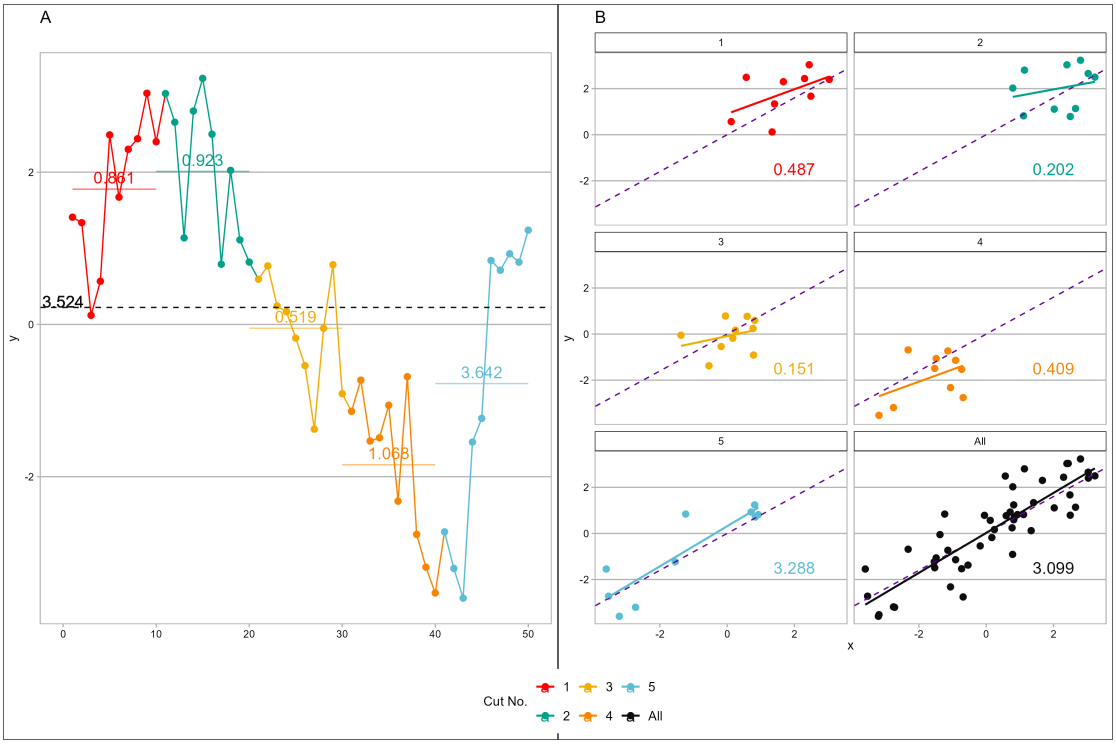


Figure 3: An AR(1) time series $y_t = 0.8y_{t-1} + \epsilon_t, \epsilon_t \sim N(0,1)$ with 50 time points. Panel A shows the trajectory of y_t with respect to time. The time series was divided into five time series of 10 consecutive time points. Five different colors are used to represent each of the time series. Solid lines and the dashed line indicate the sample mean of each of the five short and longer time series. The number above each line represents the estimated sample variance of each time series. Panel B shows the scatter plot of y_t with respect to y_{t-1} for each time series. Colored lines are the fitted regression lines and purple lines are the true regression lines. The numbers in each subfigure are the estimated sample covariance of each time series.

equation A17 in Appendix A.1):

$$E(\hat{\rho}_{OLS}) = \rho - \frac{2\rho}{T-1} + O\left(\frac{1}{(T-1)^2}\right), \tag{10}$$

where $O(\cdot)$ is the order of approximation and quantifies how fast the error of the approximation decreases as the T increases [2]. Equation 10 is derived for $\alpha = 0$, which means that the mean is known to be zero. When the mean is not known, the result becomes (Marriott and Pope, 1954):

$$E(\hat{\rho}_{OLS}) = \rho - \frac{1+3\rho}{T-1} + O\left(\frac{1}{(T-1)^2}\right). \tag{11}$$

We illustrate the consequences of this bias and its dependence on the number of available time points T for the sampling distribution of the AR effect estimator in Figure 4 for $\rho = 0.8$ and T ranging from 5 to 100. In each case, the sampling distribution is generated by simulating 1000 time series and fitting the AR(1) model to the data using the `lm(.)` function in R Core Team (2024).

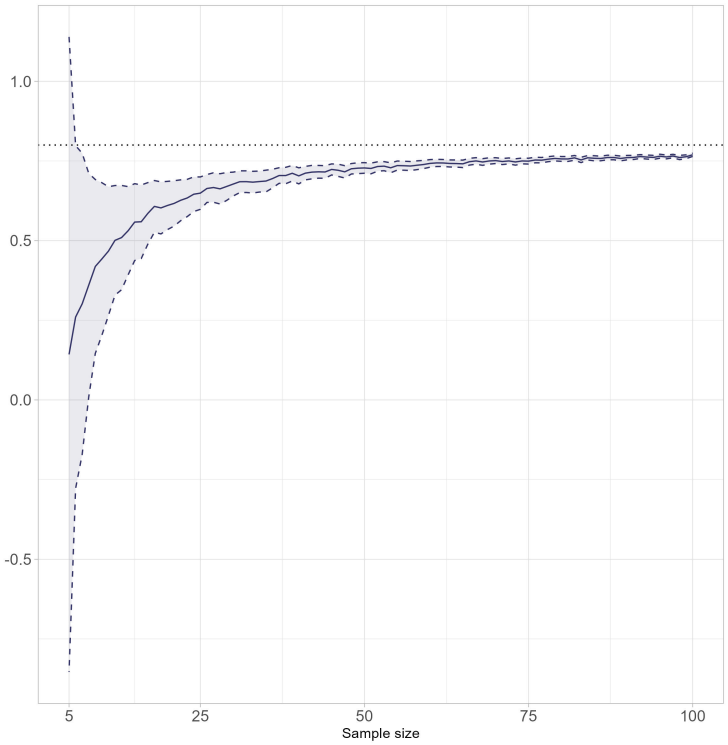


Figure 4: Average OLS estimate (with standard deviation) of AR(1) across 1000 replicates, $\rho = 0.8$ and $\alpha = 0$.

As we can observe in Figure 4, when $\rho = 0.8$, the bias of $\hat{\rho}_{OLS}$ is downward and is around -0.38 when $T = 10$. However, for $T = 100$, the bias is almost negligible. These numerical results confirm the theoretical results, showing that as the sample size increases, the bias diminishes.

3.2 Bias in the variance estimator of the OLS estimator of the AR effect

From Figure 4, we can observe that the sampling distribution of the OLS estimator seems to be asymmetric when T is small. Specifically, using a similar Taylor expansion technique that

[2] To be specific, it represents that there exists a positive real number C, such that $| E(\hat{\rho}_{OLS}) - (\rho - \frac{2\rho}{T-1}) | \leq C\frac{1}{(T-1)^2}$ as $T \to \infty$.

was used for deriving equation 8, we can obtain an approximation equation of the variance of $\hat{\rho}_{OLS}$ (see equation A24 in Appendix A.2), which is the same as the one derived by Bartlett (1946).

$$Var(\hat{\rho}_{OLS}) \approx \frac{1-\rho^2}{T-1} \tag{12}$$

The variance of $\hat{\rho}_{OLS}$ also provides information about the standard error of an estimator, which is defined as the square root of $Var(\hat{\rho}_{OLS})$. However, Huitema and McKean (1991) showed that equation 12 consistently overestimates the variance of $\hat{\rho}_{OLS}$ when $T < 100$. Thus, the corresponding standard error of $\hat{\rho}_{OLS}$ is biased when the sample size is small.

From these results, we can see that the test statistic calculated in equation 5 will be underestimated in the case of small T due to the negative bias in the numerator and the positive bias in the denominator of the T statistic, leading to conservative inference when conducting a hypothesis test on the AR effect.

3.3 Bias in the OLS estimate of the intercept

We have shown that the OLS estimator of the AR effect and its standard error are biased in small samples. This is clearly problematic if researchers want to conduct inferences on the AR effect. However, it is known that bias in $\hat{\rho}_{OLS}$ will propagate to biases in other estimators, such as the intercept parameter, to the extent that the estimators are correlated.

The relationship between the OLS estimate of α and the OLS estimate of ρ can be represented as follows (see A25 in Appendix A.3):

$$E(\hat{\alpha}_{OLS}) = (1 - \hat{\rho}_{OLS})\mu_{y_t}. \tag{13}$$

Therefore, we deduce three situations for the bias of $\hat{\alpha}_{OLS}$. When $\mu_{y_t} = 0$, $\hat{\alpha}_{OLS} = 0$, and is therefore, unbiased. However, when μ_{y_t} is positive, $\hat{\alpha}_{OLS}$ is biased in the opposite direction of the bias in $\hat{\rho}_{OLS}$. Meanwhile, when μ_{y_t} is negative, the bias in $\hat{\alpha}_{OLS}$ and the bias in $\hat{\rho}_{OLS}$ have the same direction. To conclude, in the small sample setting, the OLS estimator of α is also biased when μ_{y_t} is different from zero (Nickell, 1981).

Furthermore, the correlation between the OLS estimators of the intercept and AR effect can be proved to satisfy the following equation (see equation A31 in Appendix A.3):

$$cor(\alpha, \rho)^2 = \frac{\alpha^2}{\sigma_\epsilon^2 \frac{1-\rho}{1+\rho} + \alpha^2}. \tag{14}$$

Thus, from this formula, we observe that when $\rho \neq -1$, $cor(\hat{\alpha}_{OLS}, \hat{\rho}_{OLS})$ will always be different from zero to the extent that α is non-zero.

Figure 5 illustrates the relationship between the bias in $\hat{\rho}_{OLS}$ and $\hat{\alpha}_{OLS}$ using 1000 replicates from time series of 10 time points generated using two AR(1) models with the same AR effect 0.8 and innovation variance, and with α equal to zero or one[3].

We note that when $\alpha = 0$ (i.e., $\mu_{y_t} = 0$), the expected value of $\hat{\alpha}_{OLS}$ is equal to zero. Thus, there is no clear path in the scatter plot, such that $\hat{\alpha}_{OLS}$ takes on both positive and negative values, while the sample mean across all replicates is approximately zero. However, when $\alpha \neq 0$, we can observe that a clear linear negative relation exists between $bias(\hat{\rho}_{OLS})$ and $\hat{\alpha}_{OLS}$, and most of the points are above the true intercept (i.e., $\alpha = 1$).

4 Methods for correcting the small sample bias of the OLS estimator of the AR effect

The results presented in the previous section so far reveal that the small sample bias is not related to the estimation methods themselves but rather to the limited information available

[3]Figure 15 in Appendix B.3 displays the relationship between bias in $\hat{\rho}_{OLS}$ and $\hat{\alpha}_{OLS}$ when $T = 100$.

in short time series. In this section, we reviewed two methods for correcting the OLS estimator of the AR effect (an overview of correction methods is included in Appendix A.3). We selected these methods based on their estimation performance (Kendall, 1954; Jarocinski and Marcet, 2010; MacKinnon and Smith, 1998), their computational simplicity, as well as generalizability to VAR models (Engsted and Pedersen, 2014; Jarociński and Marcet, 2019).

4.1 Kendall's correction

A popular method to correct the OLS estimator for the AR effect was proposed by Kendall (1954) and Marriott and Pope (1954). It uses a transformation of equation 11 to derive the corrected formula:

$$\hat{\rho}_{Kendall} = \frac{T-1}{T-4}\hat{\rho}_{OLS} + \frac{1}{T-4}. \tag{15}$$

Thus, Kendall's correction gives an unbiased estimate of the AR effect to the first order:

$$E(\hat{\rho}_{Kendall}) = E\left(\frac{T-1}{T-4}\hat{\rho}_{OLS} + \frac{1}{T-4}\right) = E\left(\rho + O\left(\frac{1}{T^2}\right)\right) = \rho + O\left(\frac{1}{T^2}\right). \tag{16}$$

In addition, we can derive the variance of $\hat{\rho}_{Kendall}$ as follows:

$$Var(\hat{\rho}_{Kendall}) = Var\left(\frac{T-1}{T-4}\hat{\rho}_{OLS} + \frac{1}{T-4}\right) = \frac{(T-1)^2}{(T-4)^2}Var(\hat{\rho}_{OLS}) \tag{17}$$

We can see that Kendall's correction reduces bias, albeit at the expense of precision. Specifically, when T is small, the variance of the Kendall's corrected OLS estimator is larger than the variance of the OLS estimator by a factor of $\frac{(T-1)^2}{(T-4)^2}$ (Ledolter, 2009; MacKinnon and Smith, 1998). We note that when $T = 4$, both $\hat{\rho}_{Kendall}$ and the variance of $\hat{\rho}_{Kendall}$ is undefined. In the case of extreme small sample such as $T = 5$, $Var(\hat{\rho}_{Kendall}) = 16Var(\hat{\rho}_{OLS})$

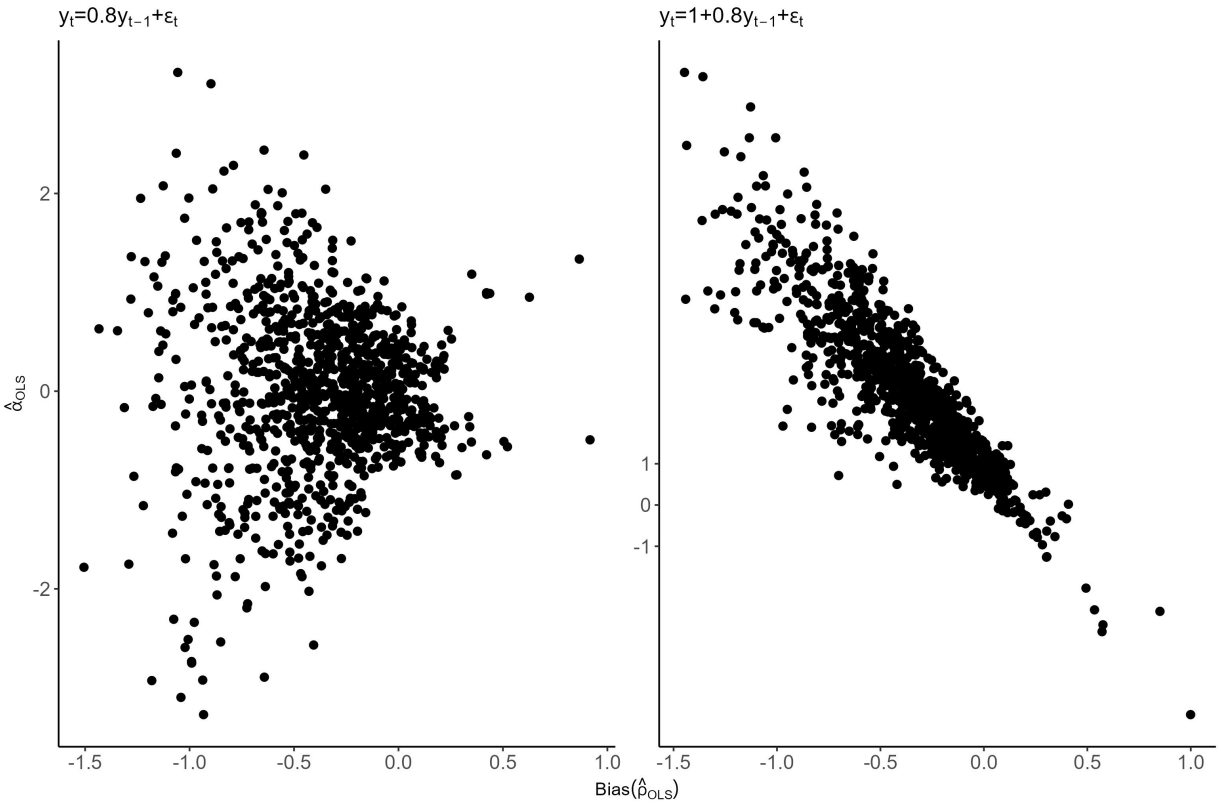


Figure 5: Bias of $\hat{\rho}_{OLS}$ versus $\hat{\alpha}_{OLS}$ across 1000 replicates, where $T = 10$, $\rho = 0.8$, $\sigma_\epsilon^2 = 1$, α equal to zero or one.

which is considerably large. Thus, Kendall's correction performs poorly when the sample size is very small. Finally, we note that equation 15 was extended to VAR(1) models by Yamamoto and Kunitomo (1984).

4.2 A correction method based on Bayesian estimation using a novel prior

As we have shown in the previous section, when α is different from zero, $\hat{\alpha}_{OLS}$ and $\hat{\rho}_{OLS}$ are correlated. Using this relation, it is reasonable to think that if we have information on plausible values of the true intercept, we could correct $\hat{\rho}_{OLS}$ to obtain a better estimator of ρ. More precisely, knowing that in the AR(1) model $\mu_{y_t} = \frac{\alpha}{1-\rho}$ and that the sample mean of y_t is an unbiased estimator of the true mean, we can use information about plausible values of α to correct the OLS estimate of the AR effect. This intuition provides the building block behind the correction method on VAR models introduced by Jarocinski and Marcet (2010). In this method, a correction for the OLS estimator of the AR effect is obtained by incorporating a prior about the intercept using a Bayesian estimation framework.

Specifically, in the AR(1) model, assuming the following prior distribution for the true intercept:

$$\alpha \sim N(\mu_\alpha, \sigma_\alpha^2), \tag{18}$$

we can derive the conditional posterior expectation of the true AR effect, which is a function of $\hat{\rho}_{OLS}$, the sample mean of y_{t-1} and y_t, T, σ_ϵ^2, and prior parameters of the intercept μ_α, σ_α^2 based on the theorem of (Jarociński and Marcet, 2019) [4]

Applying this theorem, we obtain a Bayesian corrected formula of the OLS estimator, which is given by

$$E(\rho|Y', \hat{\sigma}_\epsilon^2) = \hat{\rho}_{OLS} + (m_{01} - \hat{\rho}_{OLS} m_{10} - \mu_\alpha)\xi T m_{10} \hat{\sigma}_\epsilon^2 \sigma_\alpha^{-2}. \tag{19}$$

In this equation, $m_{10} = \frac{1}{T}\Sigma_{t=1}^{T} y_{t-1}$, $m_{01} = \frac{1}{T}\Sigma_{t=1}^{T} y_t = m_{10} + (y_T - y_0)/T$, ξ is the determinant of

$$\left(X^T X + \begin{bmatrix} \hat{\sigma}_\epsilon^2 \sigma_\alpha^{-2} & 0 \\ 0 & 0 \end{bmatrix} \right)^{-1},$$

where X is a vector including the lagged values of $Y = (y_1, ..., y_{T-1})^T$.

In Appendix B.2, we conducted a sensitivity analysis to examine how varying the values of μ_α and $\hat{\sigma}_\epsilon^2$ influences the correction. Based on the results of the sensitivity analysis, we observe that the Bayesian corrected estimates of $\hat{\rho}_{OLS}$ are sensitive to the value of μ_α. Meanwhile, varying $\hat{\sigma}_\epsilon^2$ does not change the estimation of the corrected $\hat{\rho}_{OLS}$. Thus, we keep $\hat{\sigma}_\epsilon^2 = 0.5$.

Therefore, this method requires prior information on the intercept to derive a corrected estimate of the AR effect. In a Bayesian estimation framework, when a parameter is assumed to be fixed, the prior distribution should be a point mass centered at a known value; thus, the prior probability is zero for any value different from the fixed value (Greenland, 2006). When the true intercept is known, we can set μ_α to its actual value and σ_α^2 to a value close to zero, indicating our certainty about the value of α. In this paper, we follow Jarocinski and Marcet (2010) and set σ_α^2 to 0.0001, assuming the true intercept is known from historical data.

In practice, the exact value of the intercept is typically unknown. To address this uncertainty in the Bayesian correction formula, we can set σ_α^2 to a large value (for example,

[4] The theorem states that given a Gaussian prior $g \sim \underline{N}(\mu_g, \Sigma_g)$, the posterior distribution about the parameters B, i.e., $p^g_{B|Y}$, conditioning on observing data $\overline{Y}$ is also Gaussian, and the expectation of the posterior $p^g_{B|Y}$ is

$$E_{p^g(.|\bar{Y})} = Var_{p^g(.|\bar{Y})}(B)(\Sigma_g^{-1}\mu_g + \bar{X}'\bar{Y}\Sigma_\epsilon^{-1}).$$

10,000). This approach allows us to effectively represent the uncertainty surrounding the value of α. As a result, the Bayesian correction yields a flat prior, causing equation 19 to converge to the OLS estimator of the AR effect (Jarocinski and Marcet, 2010).

Figure 6 compares the mean and standard deviation of the OLS estimate and the Bayesian correction under four different prior settings for $\rho = 0.8$, $\alpha = \{0, 1\}$, and T ranging from 5 to 100. The figure illustrates four scenarios that exhibit both correct and incorrect prior beliefs about the value of α (i.e., $\mu_\alpha = \{\alpha, 5\}$) and their associated uncertainties (i.e., $\sigma_\alpha^2 = \{0.0001, 10,000\}$).

In Figure 6, we can see that when $\alpha = 0$, the bias approaches zero as the sample size increases across different values of μ_α and σ_α^2 (refer to panel (a)). However, when $\alpha = 1$, only the scenario where the true value of α is known produces an unbiased estimate of the AR effect (refer to panel (b)).

Additionally, we observe that the Bayesian corrections with $\sigma_\alpha^2 = 10,000$ overlap with the OLS estimate in panels (a) and (b) for α values of 0 and 1. Moreover, the Bayesian correction using a precisely defined belief about the correct α (i.e., $\mu_\alpha = \alpha$ and $\sigma_\alpha^2 = 0.0001$) provides nearly unbiased estimates with a small standard deviation in both panels. Conversely, when the Bayesian correction implies a precise belief about an incorrect α (i.e., $\mu_\alpha = 5$ and $\sigma_\alpha^2 = 0.0001$), the standard deviation remains large in both panels due to the incorrect specification of μ_α.

From equation 19, we observe that the Bayesian correction adjusts the value of ρ_{OLS} based on the difference between OLS estimate of the intercept (i.e., $m_{01} - \hat{\rho}_{\text{OLS}} m_{10}$) and μ_α. This amount is scaled by the sample variance, the number of time points, the sample mean, and the ratio of the innovation variance to the prior variance of α. Therefore, when $\sigma_\alpha^2 = 0.0001$, the correction is mainly determined by the specification of μ_α.

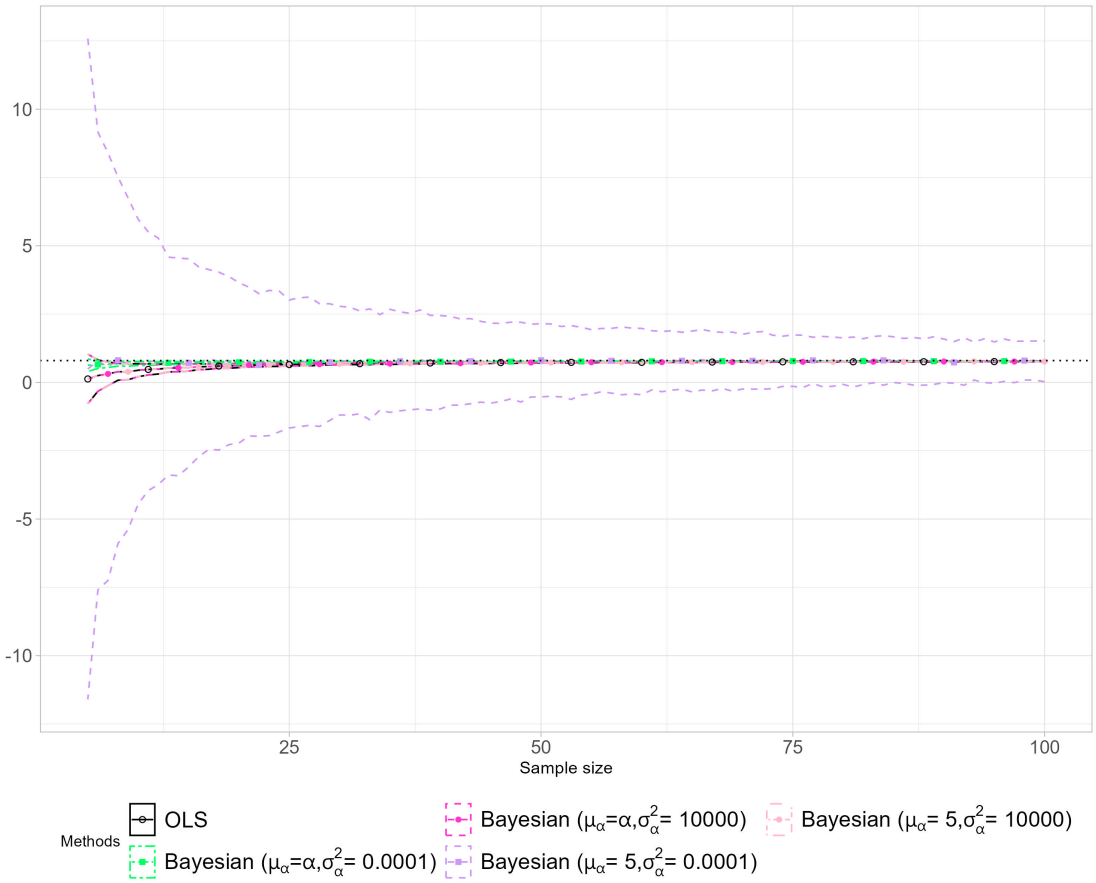
Moreover, we know that when $\alpha = 0$, the correlation between α and ρ is zero and $\mu_{y_t} = 0$. Therefore, a misspecified prior for μ_α could inflate the variance of the estimates. However, when $\alpha \neq 0$, the correlation between α and ρ is nonzero. Accordingly, in panel (b) of Figure 6, the Bayesian correction method with $\sigma_\alpha^2 = 0.0001$ and $\mu_\alpha = 5$ pushes the estimates towards lower values because of the negative correlation under this condition. Thus, using an informative prior that is misspecified can be detrimental to estimation.

Therefore, the key ingredient for obtaining an unbiased estimator of the AR effect using Bayesian correction lies in the information about the prior distribution of the parameter α.

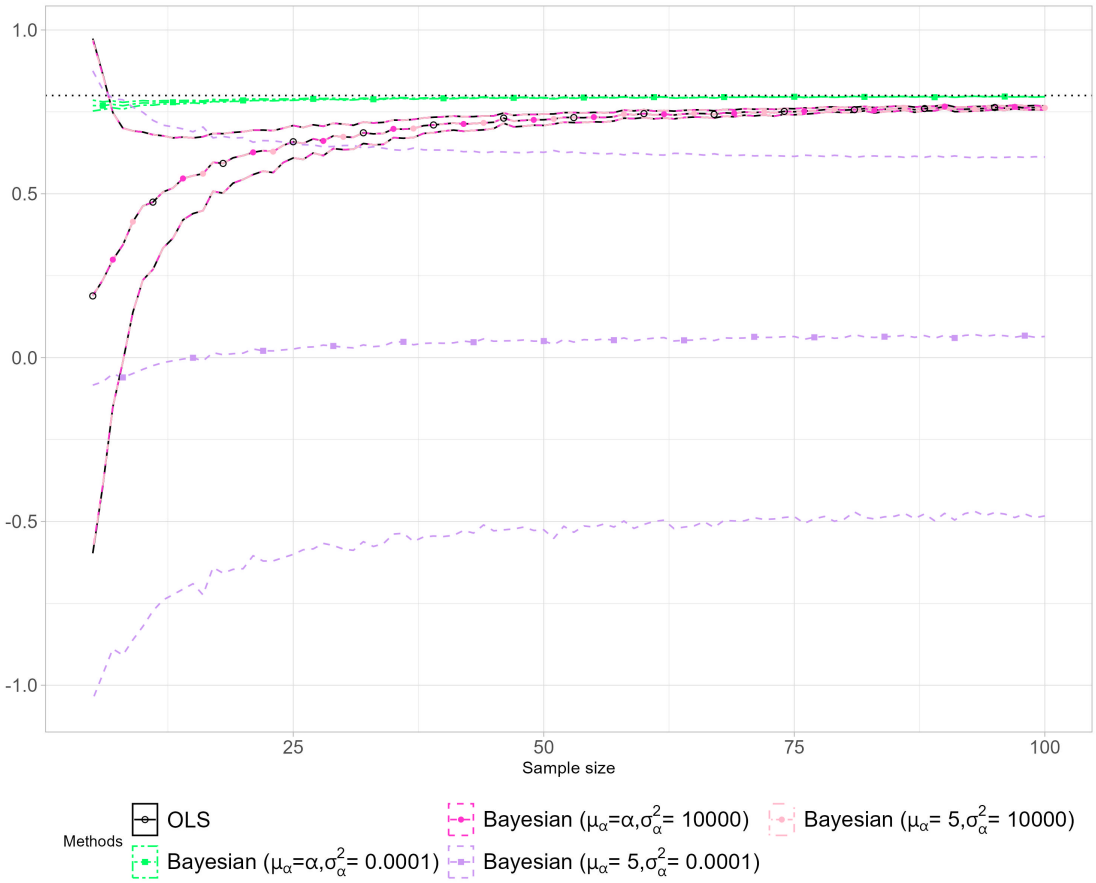
It is often impractical to assume that α is known in real-life scenarios. The question of how to obtain information about α then arises. A first possibility could be to use the OLS estimate, and as such specify that $\mu_\alpha = \hat{\alpha}_{OLS}$ in equation 19. This results in $\hat{\rho}_{OLS}$, as $\hat{\alpha}_{OLS}$ is upward biased. A second possibility could be to use a bias-corrected intercept estimate, such as $\hat{\alpha}_{Kendall}$. We will evaluate this procedure in the following section and return to the problem of specifying μ_α in the discussion. In the next section, we will assess the performance of the Bayesian correction method by using $\hat{\alpha}_{Kendall}$ to calculate μ_α in equation 19, with σ_α^2 set to 0.0001.

5 Simulation study

We conducted a simulation study to investigate the performance of the correction methods in estimating the AR effect and compare them to the performance of the OLS estimator. Specifically, we investigated the performance of the three methods in estimating the AR effect when the sample size is small in terms of bias and variance. In addition, we investigated the performance of the methods when conducting inference to test whether the AR effect is significantly different from zero (the null hypothesis: $H_0 : \rho = 0$), enabling us to evaluate the practical advantages of the correction methods despite the trade-off between bias and variance.



(a) $\alpha = 0$



(b) $\alpha = 1$

Figure 6: Mean $\pm$ standard deviation of OLS estimate and Bayesian corrected estimates of four different priors (with SD) of the AR(1) model across 1000 replicates, $\rho = 0.8$, $\sigma_\epsilon^2 = 1$. The mean is shown as the middle line, and the mean $\pm$ standard deviation are shown as the upper and lower boundary lines.

5.1 Simulation design

In the simulation study, to compare the performance of Kendall's method and the Bayesian correction method with $\hat{\alpha}_{Kendall}$ as the prior (i.e., $\mu_\alpha = \hat{\alpha}_{Kendall}$), with the OLS estimate, we systematically varied the values of the three parameters of the AR(1) model and the number of time points:

- the intercept α was set to 0, 1, 5, 10, or 50.
- the auto-regressive parameter ρ value was set to 0, .2, .4, .6, or .8.
- the variance of the innovations σ_ϵ^2 was set to 1 or 10.
- the number of time points T in each time series was set to 5, 10, 20, or 50.

Fully crossing the specification of these four parameters yielded 200 design cells. For every simulated AR(1) time series, we used a burn-in phase of 1000 time points (Bulteel et al., 2018). In each design cell, we used 1000 replicates to calculate statistics of interest: bias, mean, and standard deviation (i.e., SD). Furthermore, we compare the standard error obtained from the function `lm(.)` in R Core Team (2024), i.e., equation 6 [5], with the standard deviation of the generated replicates in each design cell, which is regarded as the true value of standard error in this simulation.

5.2 Results

5.2.1 Estimation performance

Figures 7, 8 and 10 summarize the bias, mean, and SD when $\sigma_\epsilon^2 = 1$ [6]. The simulation results, in general, conform to the discussions in the previous sections.

We observed that both bias and SD across the 1000 replicates are excessively high under the condition of extremely small time points $T = 5$. As expected, both bias and SD of the three methods decrease as the sample size increases (i.e., $T = 10, 20, 50$). To illustrate, the size of the small sample bias of the OLS estimator decreases from 10^{-1} to 10^{-2} as the sample size increases from 5 to 50. Meanwhile, SD of the OLS estimator decreases from approximately 0.5 to 0.1 as the sample size increases from 5 to 50. We found that both correction methods correct for bias when the sample size is larger than 5. To be specific, Kendall's method exhibits nearly no bias when T is 10 or larger. It is the best method with respect to bias correction among all the tested methods, albeit it had the largest SD under all the design cells. As for the Bayesian correction method, the bias and SD always lie between OLS estimators and Kendall's method when α is equal to zero or one. To be specific, it reduced bias to a certain degree when T is 10 or larger, while maintaining an acceptable SD. When α increases, the Bayesian correction method yields results similar to Kendall's method for both bias and SD, which is because the Bayesian correction method inherits the bias-variance trade-off of Kendall's method by using Kendall's intercept estimator $\hat{\alpha}_{Kendall}$ as the prior. We know the correlation between α and ρ is 0 when $\alpha = 0$. In this manner, the performance of the Bayesian correction method is not dependent on the variability of $\hat{\alpha}_{Kendall}$. However, the correlation between α and ρ is not zero if $\alpha \neq 0$. Under the circumstances, the performance of the Bayesian correction method is similar to that of Kendall's method. Therefore, the phenomenon of inheriting the performance of Kendall's method for the Bayesian correction method is not evident when $\alpha = 0, 1$. However, when α is larger than 1, the Bayesian correction method performs similarly to Kendall's method in terms of bias.

[5] In each design cell, we calculated the square root of the mean of the square standard error given by the R function for the 1000 replicates.

[6] We presented the summarized results of $\sigma_\epsilon^2 = 10$ in Appendix B.3 because varying σ_ϵ^2 did not show large differences in the simulation results.

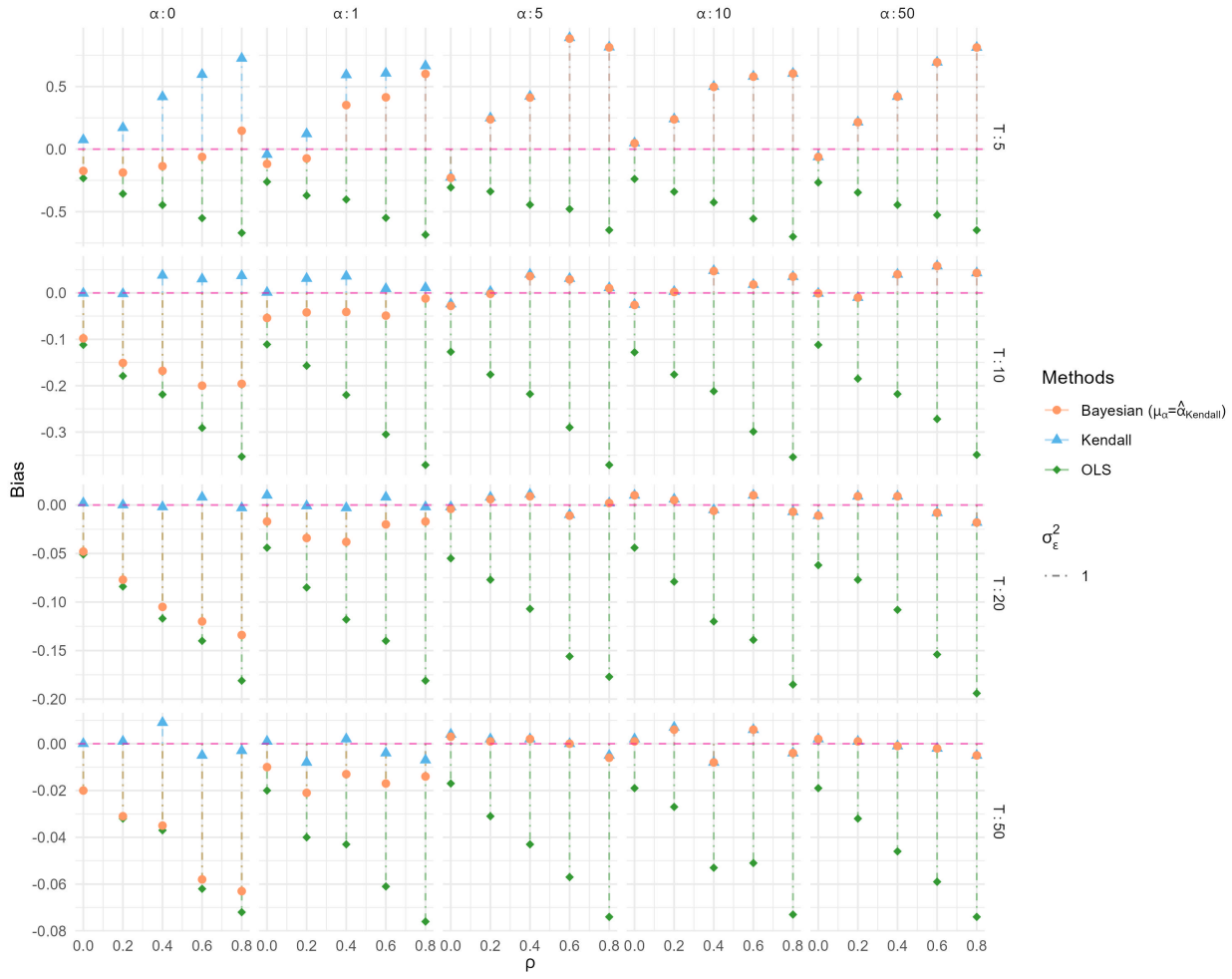


Figure 7: Simulation results. Bias of OLS estimator and corrected estimators when $\sigma^2_\epsilon = 1$ across 1000 replicates.

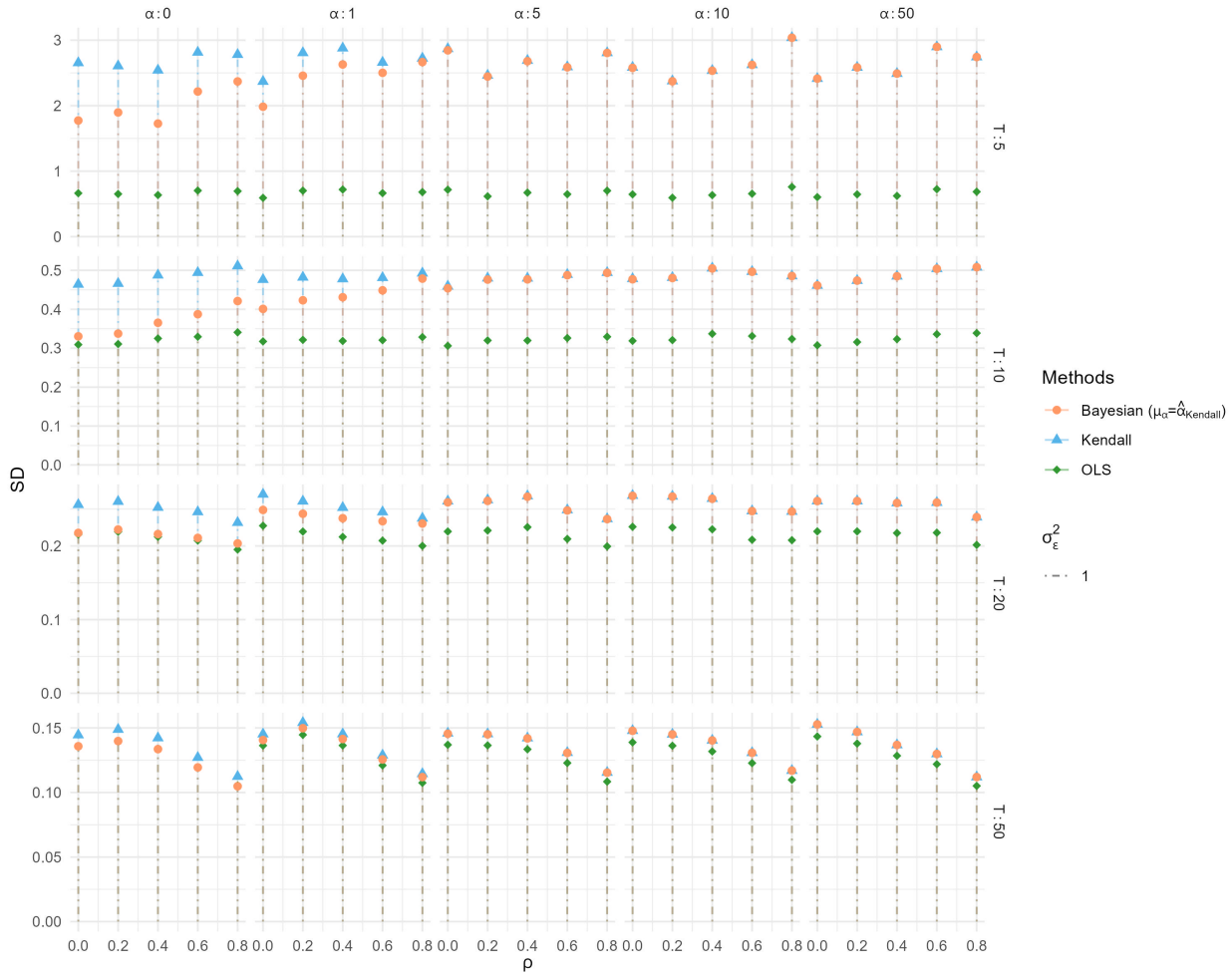


Figure 8: Simulation results. Standard deviation of OLS estimator and corrected estimators when $\sigma^2_\epsilon = 1$ across 1000 replicates.

The value of the true AR effect ρ is also an important factor influencing the small sample bias as shown in Figure 7. Equation 8 showed that the bias increases as ρ increases, when T is kept constant[7]. In contrast, we do not observe evident differences in the SD when varying ρ, although the SD of the three methods tended to be smaller when ρ increased under the condition of $T = 20$ or $T = 50$. Specifically, we observe an obvious trend of increasing bias of the OLS estimator when increasing ρ in each sub-figure of Figure 7. Unlike the OLS estimator, both correction methods did not exhibit clear patterns between bias and ρ in most sub-figures. Nevertheless, the absolute bias of Kendall's method exhibited a similar behavior to the OLS estimator when $T = 5$, i.e., the absolute bias increased as ρ increased. As for the Bayesian correction method, we observe the same trend in several cases, when $\alpha = 0$ and for the case with $\alpha = 1$ and $T = 5$.

It is noteworthy to mention that $T = 5$ is a special case among all the conditions. The bias of all the methods is extremely high except for the Bayesian correction method when $\alpha = 0$. Meanwhile, the SD of both corrections is very large; for the Kendall's method, it has been shown in the previous section that $Var(\rho^{Kendall}) = 16Var(\hat{\rho}_{OLS})$ when $T = 5$.

Regarding the Bayesian correction method, we believe its performance is primarily influenced by the fact that we use an estimator for the prior of the intercept based on Kendall's correction. Hence, the variability of the Bayesian correction method is mainly determined by the variability of Kendall's method.

In general, we observe that the OLS estimator exhibits a large bias with a small SD, while Kendall's corrected estimator sacrifices SD to reduce bias. The Bayesian correction method, however, is located between this bias-variance trade-off.

In the previous section, we demonstrated that the commonly used equation for the standard error of the OLS estimator overestimates the true standard deviation when the sample size is small, indicating a positive bias in standard error estimation in small samples. Figure 9 compares the commonly used standard error of $\hat{\rho}_{OLS}$ (i.e., equation 6) and the standard deviation of the OLS estimates over simulation replications. We observed that differences between the two statistics are evident for $T = 5, 10$, indicating a small positive bias in the standard error of the OLS estimator. This difference decreases when T increases. Besides, we noticed that this difference might be related to the value of ρ. When T is sufficiently large, we know that the estimated standard error decreases when ρ increases from the approximation equation of the variance of $\hat{\rho}_{OLS}$, i.e., equation 12. Thus, our observations in Figure 9 are consistent with equation 12: when T is fixed and sufficiently large, the standard error of $\hat{\rho}_{OLS}$ decreases when ρ increases. Regarding the intercept, we did not observe a noticeable effect on this difference as α varied.

5.2.2 Inferential performance

In addition, we calculated the Type I error rate and power to detect whether the AR effect differs from zero (i.e., $H_0 : \rho = 0$) as the proportion of significant replications across all replications in each design cell. To be specific, with respect to the OLS estimator, we performed hypothesis testing based on the p-value of the AR(1) model result (i.e., the function `lm(.)` from R Core Team (2024)).

We employed the permutation test to examine the null hypothesis $H_0 : \rho = 0$ concerning the two correction methods. The significance level for our hypothesis testing was set at 5%. It is important to note that conventional inference methods do not apply to these correction techniques. This limitation arises because the analytical formulas for the variance of the corrected estimates are derived from the OLS estimator. Moreover, the analytical formula for the variance of the OLS estimator, as shown in equation 6, is biased. Consequently, the inferential results may not be reliable if we directly substitute ρ with $\hat{\rho}_{OLS}$ in the variance

[7]We can also see this from the approximation formula of $\hat{\rho}_{OLS}$ in the work of Marriott and Pope (1954), or see Figure 1 in the Appendix.

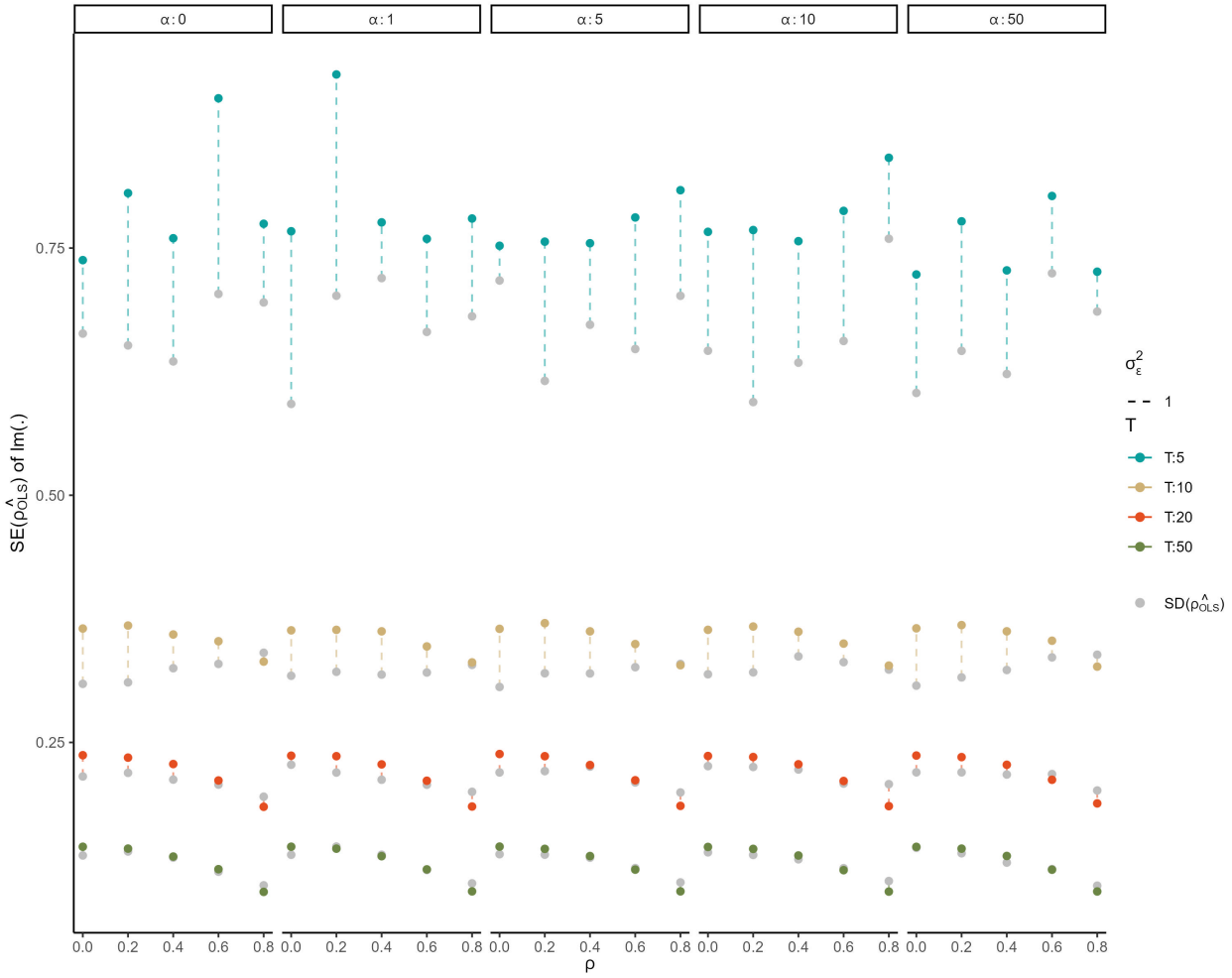


Figure 9: Simulation results. Comparison between standard error of OLS estimator from the function `lm(.)` and the standard deviation of $\hat{\rho}_{OLS}$ across 1000 replicates when $\sigma^2_\epsilon = 1$.

equation, especially when the sample size is small (Huitema and McKean, 1991)[8]. Therefore, to perform inference on the null hypothesis $H_0 : \rho = 0$, we applied the permutation test, which imposes no assumption on the distribution of the errors in the AR(1) model (Pitman, 1937). The idea behind the permutation test is to completely break down the data distribution by resampling without replacement. Thus, if an effect exists in the data, the probability of having an effect of equal or larger size in a broken and randomly resampled data set will be very small.

To be specific, for every generated AR(1) time series, e.g., $y_1, \ldots, y_T$, we repeatedly sampled 1000 new versions of this dataset without replacement, e.g., $y_1^*, \ldots, y_T^*$. The lagged variable in the new dataset is also included as the predictors, e.g., $NA, y_1^*, \ldots, y_{T-1}^*$. Then, for each new permuted version of the dataset, we calculated the corresponding corrected estimators using the correction methods. Taking the 1000 values derived for each correction method as the empirical distribution of ρ under the null hypothesis $H_0 : \rho = 0$, we could calculate the p-value using the quantile of the estimated corrected AR effect based on the original data. Using this $p-$value, we performed a hypothesis test: we reject H_0 if the $p-$value is smaller than 0.05, otherwise, we do not reject H_0.



Figure 10: The first row of the sub figures presents the estimated Type I error rate and the second to fourth rows of the sub figures present the estimated power of the OLS estimator and corrected estimators when $\sigma_\epsilon^2 = 1$. Pink lines display the nominal Type I error rate and statistical power of simple linear regression in each design cell.

Figure 10 shows the type I error rate and power of the three estimators of the AR effect when testing if the AR effect is different from zero. Regarding the recovery of the nominal

[8]Several approximation formulas of the empirical variance of estimator in the AR model have been derived, see, for example, works of Bartlett (1946); Marriott and Pope (1954); Moran (1948).

type I error rate of the OLS estimator using the `lm(.)` function, we observe that it is always below the nominal level 5%, which means it is always conservative. We observe that the performance of the OLS estimator on type I error recovery is influenced by the value of the intercept α and σ_ϵ^2. When $T = 20, 50$, we observed that the type I error rate increases when α increases in Figure 10. We notice that for $T = 20, 50$ 10 , the type I error rate varies more when σ_ϵ^2 is 10 in comparison to the case in which σ_ϵ^2 is 1 (the results for σ_ϵ^2 equal to 10 are included in the Appendix B.3).

We observed that the results of the estimated type I error of Kendall's correction fluctuate above the nominal level 5% when $T \leq 20$. Regarding the Bayesian correction method, the type I error rate is similar and slightly more stable, around 5% in most conditions, when compared to Kendall's method. When $T = 50$, the type I error rates of all the correction methods are either below or around 5%.

In terms of power, both correction methods yield more accurate power estimates compared to the OLS estimator. As a benchmark, we included the estimated power for testing the significance of a slope in a simple linear regression (i.e., represented by the horizontal line in Figure 10 when the AR effect is non-zero). We expect that the estimated power for testing if the slope is significantly different from zero to be higher than the power obtained when conducting a hypothesis test related to the AR effect. The power of the correction methods reaches the power level of a simple linear regression under certain conditions, specifically when $\rho = 0.2, 0.4$ and $T = 50$, as well as for several conditions when $\rho = 0.6, 0.8$. Additionally, we found that the estimated power of the Bayesian correction method approaches that of Kendall's method as α increases. However, both correction methods become overly confident regarding type I error when α increases.

6 Discussion

In this paper, we reviewed the causes and consequences of using small samples when estimating the AR effect in stationary AR(1) models using OLS. While this problem has been extensively studied in the statistical literature, we offered a new perspective by describing the mechanisms that lead to the small sample bias when estimating the AR effect. Furthermore, we show that small samples also affect the estimation of other key sample statistics, such as the sample variance and autocovariance of y_t, the standard error of the OLS estimator, and the OLS intercept estimate. Additionally, we focused on the inferential performance of the OLS estimator of the AR effect and demonstrated how small samples ultimately result in conservative inference. Finally, we explored potential correction methods to address the small sample bias and conducted a simulation study to compare the performance of OLS with these correction methods.

6.1 Our findings

As we have shown, the small sample bias in the estimation of the AR effect arises from the violation of the exogeneity assumption ($E(\epsilon_t \mid y_1, ..., y_T) = 0$). This assumption is crucial not only for OLS but also for other methods. Krone et al. (2017) compared state-of-the-art frequentist and Bayesian methods for estimating the AR effect in short time series and found that small sample bias persists across the different estimation approaches. In essence, this bias in estimating the AR effect stems from limited information. Consequently, researchers cannot avoid this issue by merely using more sophisticated estimation techniques that do not rely on informative prior knowledge.

The analytic and simulation results also show that addressing the small sample bias using non-informative correction methods always entails a bias-variance trade-off, that is, reducing bias may increase estimation uncertainty. Although the correction formulas presented in this paper can yield nearly unbiased estimates of the AR effect when $T > 5$, they always imply an increase in variance compared to the OLS estimator. Thus, from a practical standpoint, the correction on the AR effect might not offer additional gains for researchers compared to

the OLS estimator due to the bias-variance trade-off.

When examining the relationship between the estimated intercept and the AR effect in an OLS regression, a more precise method that incorporates the known α to correct for the AR effect may be available via the Bayesian correction formula. This is, for example, applied in economic research, where priors on the initial growth rate can be translated into information about the distribution of the intercept, which can be used to correct for bias in the AR effect estimator (Jarociński and Marcet, 2019). However, because prior information on the initial growth rates of psychological variables is rarely discussed in psychological applications, it is challenging to see how such priors can be effectively employed in these contexts. Nevertheless, if the intercept (or the mean) of the AR process is assumed to be known in a psychological study (e.g., the mean is assumed to be zero), researchers can leverage this information to obtain a more accurate AR estimate.

Due to the challenges of compensating for limited information and the inherent unreliability of resulting estimates, we stress the need for caution when conducting research with small sample sizes, particularly those with fewer than 50 time points. When the sample size is 50 or greater, we recommend using correction methods such as Kendall's formula, which ensure an appropriate Type I error rate and provide higher power than OLS.

If collecting large samples is not feasible, the choice of estimator for the AR effect (whether OLS, another estimator, or a corrected estimator) should base on the study's objectives. If the goal is accurate point estimation, correction methods (e.g., Kendall's method) could be used. Conversely, if the goal is to conduct inference with a limited sample size ($T < 50$), relying on an uncorrected AR effect estimator (e.g., the OLS estimator) is appropriate, as it preserves a low Type I error rate, albeit at the expense of being conservative and having low power.

On the other hand, correction methods may fail to maintain an appropriate Type I error rate due to the bias-variance trade-off when the sample size is limited ($T < 50$). Furthermore, corrected estimators may not be appropriate if the goal is to detect significant between-group differences in the AR effect. When the true AR effect is assumed to be fixed within each group, the corrected estimator in each group is essentially a linear transformation of the original estimates. As a result, the between-group difference in the corrected AR effects remains a linear transformation of the original difference, and the overall conclusions are unlikely to change.

6.2 Directions for future research

Future research in the small sample bias area could focus on extensions to VAR models and multilevel models. Regarding VAR, both correction methods discussed in the paper could be extended to AR models, where small sample estimation can also be problematic (Jarocinski and Marcet, 2010; Yamamoto and Kunitomo, 1984). Moreover, Bhansali (1981) obtained the bias formula of OLS estimates in models of arbitrary order with known mean and unknown true order, which reduces to Kendall's formula in the AR(1) model (Shaman and Stine, 1988). However, the performance of these correction methods should again be thoroughly investigated in a simulation study with a setup similar to the one reported in this paper.

Regarding multilevel models, we examined the consequences of small sample bias and explored correction methods for the person-specific AR(1) model. Though beyond the scope of this paper, one might wonder about the potential benefit of using multilevel AR(1) models, which becomes especially relevant when one is interested in predicting the AR effect with a covariate or in using the AR effect to predict an outcome variable. Multilevel AR(1) models have indeed been used to estimate person-specific AR effects (see e.g., Lafit et al. (2022)), where the model is fit to time series data from multiple participants simultaneously. The person-specific AR effects are obtained by summing the fixed AR effect and each participant's estimated random AR effect. However, several considerations must be taken into account

when using multilevel AR(1) models to estimate person-specific AR effects. First, multilevel models come with strict assumptions regarding the distribution of the random AR effects, which is usually assumed to be normal (see Gistelinck et al. (2021); Krone et al. (2016); Lafit et al. (2022)). For example, focusing on the VAR(1) model, Lafit et al. (2022) showed that when the length of the time series is 50, multilevel VAR(1) models can accurately estimate the random autoregressive and cross-regressive effects when differences in the distribution of the person-specific auto- and cross-regressive effects when they follow a normal or a uniform distribution. In contrast, when the AR effects across participants are expected to follow a mixture distribution, mixture multilevel AR(1) models may be preferred (see e.g., Ernst et al. (2024b); Lafit et al. (2022).

Second, including participants with time series of varying lengths can lead to an imbalanced dataset. Eager and Roy (2017) demonstrated that such imbalanced data can cause significant convergence problems when fitting linear and generalized multilevel models. Further research is needed to investigate the effects of imbalanced time series data on the estimation of multilevel AR(1) models.

Third, when fitting multilevel AR(1) models with mixed-effects methods, it is crucial to decide whether to person-mean center each participant's time series. Hamaker and Grasman (2015) found that person-mean centering can lead to a downward bias in the AR parameter estimate, and this bias can be worse for short time series. We note that estimation of multilevel AR(1) models using structural equation models (SEM) offers an advantage as it does not require person-mean centering each participant's time series before estimating the model. In this respect, Gistelinck et al. (2021) compared several structural equation models for multilevel AR(1) models. They found that the non-centering approach, in which the initial time point is allowed to be correlated with the random intercept, performs best in small samples. This aligns with our discussion of the correlation between OLS estimates of α and ρ and the correction method proposed by Jarocinski and Marcet (2010). These results show that incorporating intercept information, as we discussed in this paper, could potentially improve model estimation in multilevel AR(1) models in small sample settings.

Finally, many psychological applications involve time series data with missing values, which can reduce the available information for estimating the AR effect and thus worsen the small sample bias issue. Revol et al. (2024) studied the impact of missing data mechanisms on the estimation performance of person-specific AR(1) and multilevel AR(1) models. Their simulation study revealed that when data is assumed to be missing completely at random, both the person-specific AR effect and the fixed effect in multilevel models are slightly underestimated. However, when data are not missing at random, the underestimation of both the person-specific and fixed AR effects is more pronounced.

7 Conclusion

In this study, we examined the causes and consequences of using small samples when estimating the AR effect using OLS. We also compare the performance of the OLS estimator with two correction methods. Our results indicate that small sample biases are present, and can have severe consequences for the validity of inferences drawn from the parameter estimates. Furthermore, Kendall's bias correction method has limited practical applicability in very small samples due to the increase in estimator variance associated with the bias correction method.

The ultimate solution to the small-sample bias lies in incorporating more information during estimation, either by increasing the sample size or by leveraging the relationship between the AR(1) model's parameters using Bayesian techniques.

Conflicts of Interest

The authors declare no conflicts of interest concerning the authorship or the publication of this article.

Supplemental Material

The code for this paper is available at `https://gitlab.kuleuven.be/u0167074/small-sample-ar-models.git`.

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A Appendices

A.1 Derivation of the OLS estimator regarding the exogeneity term

Given the moving average representation of the AR(1) model: $y_t = \frac{\alpha}{1-\rho} + \sum_{j=0}^{\infty} \rho^j \epsilon_{t-j}$, we can derive the covariance between innovation ϵ_t and current and future values of the process

as follows:

$$\begin{aligned} cov(\epsilon_t, y_{t+s}) &= cov\left(\epsilon_t, \frac{\alpha}{1-\rho} + \sum_{j=0}^{\infty} \rho^j \epsilon_{t+s-j}\right) \\ &= cov(\epsilon_t, \rho^s \epsilon_{t+s-s}) \\ &= cov(\epsilon_t, \rho^s \epsilon_t) \\ &= \rho^s cov(\epsilon_t, \epsilon_t) \\ &= \rho^s \sigma_\epsilon^2. \end{aligned} \tag{A1}$$

This shows that strict exogeneity fails to hold for the AR(1) model. Indeed, ϵ_t is correlated with $y_t, \ldots, y_T$. To show that this causes bias in the OLS estimator, notice that the OLS estimator (Equation 3) can be rewritten as:

$$\begin{aligned} \hat{\rho}_{OLS} &= \frac{(T-1)\sum_2^T y_t y_{t-1} - \sum_2^T y_{t-1} \sum_2^T y_t}{(T-1)\sum_2^T y_{t-1}^2 - (\sum_{t=2}^T y_{t-1})^2} \\ &= \frac{(T-1)\sum_2^T (\alpha + \rho y_{t-1} + \epsilon_t) y_{t-1} - \sum_2^T y_{t-1} \sum_2^T (\alpha + \rho y_{t-1} + \epsilon_t)}{(T-1)\sum_2^T y_{t-1}^2 - (\sum_{t=2}^T y_{t-1})^2} \\ &= \frac{(T-1)\alpha \sum_2^T y_{t-1} + (T-1)\rho \sum_2^T y_{t-1}^2 + (T-1)\sum_2^T \epsilon_t y_{t-1} - (T-1)\alpha \sum_2^T y_{t-1} - \rho(\sum_2^T y_{t-1})^2 - \sum_{t=2}^T y_{t-1} \sum_2^T \epsilon_t}{(T-1)\sum_2^T y_{t-1}^2 - (\sum_{t=2}^T y_{t-1})^2} \\ &= \rho + \frac{(T-1)\sum_2^T \epsilon_t y_{t-1} - \sum_{t=2}^T y_{t-1} \sum_2^T \epsilon_t}{(T-1)\sum_2^T y_{t-1}^2 - (\sum_{t=2}^T y_{t-1})^2}. \end{aligned} \tag{A2}$$

In equation $1, \epsilon_t \overset{\text{i.i.d}}{\sim} N(0, \sigma_\epsilon^2)$ provides that $E(\epsilon_t) = 0$ and $E(\sum_2^T \epsilon_t) = \sum_2^T E(\epsilon_t) = 0$. Knowing that $E(AB) = E(A)E(B) + cov(A, B)$ (Rice, 2006), the expected value of a combination of ϵ_t and $y_1, ..., y_T$ is $E(\epsilon_t f(y_1, ..., y_T)) = E(\epsilon_t)E(f(y_1, ..., y_T)) + cov(\epsilon_t, f(y_1, ..., y_T)) = cov(\epsilon_t, f(y_1, ..., y_T))$ where $f(y_1, ..., y_T)$ denotes a function of $y_1, ..., y_T$. If the strict exogeneity assumption holds, ϵ_t is independent of $y_1, ..., y_T$ which means that $cov(\epsilon_t, f(y_1, ..., y_T)) = 0$. Thus, $E(\epsilon_t f(y_1, ..., y_T)) = 0$, and the OLS estimator is unbiased, $E(\hat{\rho}_{OLS}) = \rho$.

However, the strict exogeneity assumption is violated for the AR(1) model as mentioned above. Thus,

$$E(\epsilon_t f(y_1, ..., y_{t-1})) = 0, \tag{A3}$$

but

$$E(\epsilon_t f(y_t, ..., y_T)) \neq 0. \tag{A4}$$

The expected value of the OLS estimator is in fact:

$$\begin{aligned} E(\hat{\rho}_{OLS}) &= E\left(\rho + \frac{(T-1)\sum_2^T \epsilon_t y_{t-1} - \sum_{t=2}^T y_{t-1} \sum_2^T \epsilon_t}{(T-1)\sum_2^T y_{t-1}^2 - (\sum_{t=2}^T y_{t-1})^2}\right) \\ &= \rho + E\left(\frac{(T-1)\sum_2^T \epsilon_t y_{t-1}}{(T-1)\sum_2^T y_{t-1}^2 - (\sum_{t=2}^T y_{t-1})^2}\right) - E\left(\frac{\sum_{t2}^T y_{t-1} \sum_2^T \epsilon_t}{(T-1)\sum_2^T y_{t-1}^2 - (\sum_{t=2}^T y_{t-1})^2}\right) \\ &= \rho + (T-1)\sum_2^T E\left(\epsilon_t \frac{y_{t-1}}{(T-1)\sum_{\tau=2}^T y_{\tau-1}^2 - (\sum_{\tau=2}^T y_{\tau-1})^2}\right) - \sum_2^T E\left(\epsilon_t \frac{\sum_{\tau=2}^T y_{\tau-1}}{(T-1)y_{t-1}^2 - (\sum_{\tau=2}^T y_{\tau-1})^2}\right) \\ &= \rho + (T-1)\sum_2^T cov\left(\epsilon_t, \frac{y_{t-1}}{(T-1)\sum_{\tau=2}^T y_{\tau-1}^2 - (\sum_{\tau=2}^T y_{\tau-1})^2}\right) - \sum_2^T cov\left(\epsilon_t, \frac{\sum_{\tau=2}^T y_{\tau-1}}{(T-1)y_{t-1}^2 - (\sum_{\tau=2}^T y_{\tau-1})^2}\right) \end{aligned} \tag{A5}$$

Moreover when $\alpha = 0$, this equation is simplified to

$$\begin{aligned} E(\hat{\rho}_{OLS}) &= E\left(\rho + \frac{\Sigma_2^T y_{t-1}\epsilon_t}{\Sigma_2^T y_{t-1}^2}\right) \\ &= \rho + E\left(\frac{\Sigma_2^T y_{t-1}\epsilon_t}{\Sigma_2^T y_{t-1}^2}\right) \\ &= \rho + \Sigma_2^T E\left(\frac{y_{t-1}\epsilon_t}{\Sigma_{\tau=2}^T y_{\tau-1}^2}\right) \end{aligned} \tag{A6}$$

We now give a proof that the approximation of equation A6 converges to ρ as $T \to \infty$. Applying the Taylor expansion for the moments of functions of random variables to the first derivative,

$$E(\frac{X}{Y}) \approx \frac{E(X)}{E(Y)} - \frac{cov(X,Y)}{E(Y)^2} + \frac{E(X)}{E^3(Y)} var(Y), \tag{A7}$$

and combining equation A3 and equation A4, we derive that

$$\begin{aligned} E\left(\frac{y_{t-1}\epsilon_t}{\Sigma_{\tau=2}^T y_{\tau-1}^2}\right) &\approx \frac{E(y_{t-1}\epsilon_t)}{E(\Sigma_{\tau=2}^T y_{\tau-1}^2)} - \frac{cov(y_{t-1}\epsilon_t, \Sigma_{\tau=2}^T y_{\tau-1}^2)}{E(\Sigma_{\tau=2}^T y_{\tau-1}^2)^2} + \frac{E(y_{t-1}\epsilon_t)}{E(\Sigma_{\tau=2}^T y_{\tau-1}^2)^3} var(\Sigma_{\tau=2}^T y_{\tau-1}^2) \\ &= -\frac{cov(y_{t-1}\epsilon_t, \Sigma_{\tau=2}^T y_{\tau-1}^2)}{E(\Sigma_{\tau=2}^T y_{\tau-1}^2)^2} \\ &= -\frac{E(y_{t-1}\epsilon_t \Sigma_{\tau=2}^T y_{\tau-1}^2) - E(y_{t-1}\epsilon_t)E(\Sigma_{\tau=2}^T y_{\tau-1}^2)}{E(\Sigma_{\tau=2}^T y_{\tau-1}^2)^2} \\ &= -\frac{E(y_{t-1}\epsilon_t \Sigma_{\tau=2}^T y_{\tau-1}^2)}{E(\Sigma_{\tau=2}^T y_{\tau-1}^2)^2} \\ &= -\frac{E(y_{t-1}\epsilon_t (y_1^2 + \cdots + y_{T-1}^2)}{E(\Sigma_{\tau=2}^T y_{\tau-1}^2)^2} \\ &= -\frac{E(y_{t-1}\epsilon_t (y_t^2 + \cdots + y_{T-1}^2))}{E(\Sigma_{\tau=2}^T y_{\tau-1}^2)^2} \\ &= -\frac{E(y_{t-1}\epsilon_t y_t^2) + \cdots + E(y_{t-1}\epsilon_t y_T^2)}{E(\Sigma_{\tau=2}^T y_{\tau-1}^2)^2}. \end{aligned} \tag{A8}$$

Using the moving average representation of AR(1) process, $y_t \sim N(\frac{\alpha}{1-\rho}, \frac{\sigma_\epsilon^2}{1-\rho^2})$ (here, $y_t \sim N(0, \frac{\sigma_\epsilon^2}{1-\rho^2})$ because $\alpha = 0$) and $\epsilon_t \overset{\text{i.i.d}}{\sim} N(0, \sigma_\epsilon^2)$, we could further derive that

$$\begin{aligned} E(y_{t-1}\epsilon_t y_t^2) &= E(y_{t-1}\epsilon_t(\rho y_{t-1} + \epsilon_t)^2) \\ &= E(y_{t-1}\epsilon_t(\rho^2 y_{t-1}^2 + 2\rho y_{t-1}\epsilon_t + \epsilon_t^2)) \\ &= \rho^2 E(y_{t-1}^3\epsilon_t) + 2\rho E(y_{t-1}^2\epsilon_t^2) + E(y_{t-1}\epsilon_t^3) \\ &= \rho^2 E(y_{t-1}^3)E(\epsilon_t) + 2\rho E(y_{t-1}^2)E(\epsilon_t^2) + E(y_{t-1})E(\epsilon_t^3) \\ &= \frac{2\rho(\sigma_\epsilon^2)^2}{1-\rho^2} \end{aligned} \tag{A9}$$

Similarly, we could derive that, for a given integer $k \geq 0$,

$$E(y_{t-1}\epsilon_t y_{t+k}^2) = \frac{2\rho^{2k+1}(\sigma_\epsilon^2)^2}{1-\rho^2}. \tag{A10}$$

Therefore, equation A8 becomes

$$
\begin{aligned}
E\left(\frac{y_{t-1}\epsilon_t}{\Sigma_{\tau=2}^{T} y_{\tau-1}^2}\right) &\approx -\frac{E(y_{t-1}\epsilon_t y_t^2)+\cdots+E(y_{t-1}\epsilon_t y_T^2)}{E(\Sigma_{\tau=2}^{T} y_{\tau-1}^2)^2} \\
&= -\frac{2(\rho+\cdots+\rho^{2(T-t)+1})(\sigma_\epsilon^2)^2}{(1-\rho^2)E(\Sigma_{\tau=2}^{T} y_{\tau-1}^2)^2}.
\end{aligned}
\tag{A11}
$$

Equation A6 could be approximated as:

$$
\begin{aligned}
E(\hat{\rho}_{OLS}) &= \rho + \sum_{2}^{T} E\left(\frac{y_{t-1}\epsilon_t}{\Sigma_{\tau=2}^{T} y_{\tau-1}^2}\right) \\
&\approx \rho - \sum_{2}^{T} \frac{2(\rho+\cdots+\rho^{2(T-t)+1})(\sigma_\epsilon^2)^2}{(1-\rho^2)E(\Sigma_{\tau=2}^{T} y_{\tau-1}^2)^2} \\
&= \rho - \frac{2[\rho+\cdots+\rho^{2T-3}+\cdots+(\rho)](\sigma_\epsilon^2)^2}{(1-\rho^2)E(\Sigma_{\tau=2}^{T} y_{\tau-1}^2)^2} \\
&= \rho - \frac{2[(T-1)\rho+(T-2)\rho^3+\cdots+\rho^{2T-3})](\sigma_\epsilon^2)^2}{(1-\rho^2)E(\Sigma_{\tau=2}^{T} y_{\tau-1}^2)^2}
\end{aligned}
\tag{A12}
$$

We know $E(\Sigma_{\tau=2}^{T} y_{\tau-1}^2)^2 = (\Sigma_2^T E(y_{\tau-1}^2))^2 = (\frac{(T-1)\sigma_\epsilon^2}{1-\rho^2})^2$. Thus, the expected value of the OLS estimator to order T^{-1} should be

$$
\begin{aligned}
E(\hat{\rho}_{OLS}) &\approx \rho - \frac{2[(T-1)\rho+(T-2)\rho^3+\cdots+\rho^{2T-3})](\sigma_\epsilon^2)^2}{(1-\rho^2)(\frac{(T-1)\sigma_\epsilon^2}{1-\rho^2})^2} \\
&= \rho - \frac{2(1-\rho^2)\rho^{2T}\sum_{k=1}^{T-1} k\rho^{-2k-1}}{(T-1)^2} \\
&= \rho - \frac{2(1-\rho^2)\rho^{2T-3}}{(1-\rho^{-2})^2(T-1)^2} \quad as \quad T\to\infty,
\end{aligned}
\tag{A13}
$$

See equation 8 in the main text.
Since $\rho \in (-1,1)$, $\rho^{2T} \to 0$, term

$$
\frac{2(1-\rho^2)\rho^{2T-3}}{(1-\rho^{-2})^2(T-1)^2} \to 0 \quad as \quad T\to\infty. \tag{A14}
$$

Therefore, we show that (see equation 9 in the main text)

$$
E(\hat{\rho}_{OLS}) \approx \rho \quad as \quad T\to\infty. \tag{A15}
$$

Moreover, as equation A7 is an approximation to the first derivative, we can write equation A13 as

$$
E(\hat{\rho}_{OLS}) = \rho - \frac{2(1-\rho^2)\rho^{2T}\sum_{k=1}^{T-1} k\rho^{-2k-1}}{(T-1)^2} + O\left(\frac{1}{(T-1)^2}\right). \tag{A16}
$$

Thus, combining with the Taylor series, we can show that:

$$
\begin{aligned}
E(\hat{\rho}_{OLS}) &= \rho - \frac{2(1-\rho^2)\rho^{2T}\sum_{k=1}^{T-1} k\rho^{-2k-1}}{(T-1)^2} + O\left(\frac{1}{(T-1)^2}\right) \\
&= \rho - \frac{2(1-\rho^2)\sum_{k=1}^{T-1}(T-k)\rho^{2k-1}}{(T-1)^2} + O\left(\frac{1}{(T-1)^2}\right) \\
&= \rho - \frac{2(1-\rho^2)[(T-1)\sum_{k=1}^{T-1}\rho^{2k-1} + \sum_{k=1}^{T-1}(k-1)\rho^{2k-1}]}{(T-1)^2} + O\left(\frac{1}{(T-1)^2}\right) \\
&= \rho - \frac{2(1-\rho^2)[(T-1)\sum_{k=1}^{T-1}\rho^{2k-1} + \sum_{k=1}^{T-1}(k-1)\rho^{2k-1}]}{(T-1)^2} + O\left(\frac{1}{(T-1)^2}\right) \\
&= \rho - \frac{2(1-\rho^2)\sum_{k=1}^{T-1}\rho^{2k-1}}{T-1} + \frac{2(1-\rho^2)\sum_{k=2}^{T-1}(k-1)\rho^{2k-1}}{(T-1)^2} + O\left(\frac{1}{(T-1)^2}\right) \\
&= \rho - \frac{2(1-\rho^2)\frac{\rho}{1-\rho^2}}{T-1} + \frac{2(1-\rho^2)\frac{\rho^3}{(1-\rho^2)^2}}{(T-1)^2} O\left(\frac{1}{(T-1)^2}\right) \quad as \quad T \to \infty \\
&= \rho - \frac{2\rho}{T-1} + \frac{2\rho^3}{(T-1)^2(1-\rho^2)} + O\left(\frac{1}{(T-1)^2}\right) \\
&= \rho - \frac{2\rho}{T-1} + O\left(\frac{1}{(T-1)^2}\right).
\end{aligned}
\tag{A17}
$$

Equation A17 conforms to formula (4.08) in the paper of Marriott and Pope (1954).

A.2 Approximated variance formula for $\hat{\rho}_{OLS}$

Applying a similar Taylor expansion as in the previous section, we can also obtain an approximation of the variance of $\hat{\rho}_{OLS}$. We begin with the equation in equation A6,

$$
\hat{\rho}_{OLS} = \rho + \frac{\Sigma_2^T y_{t-1}\epsilon_t}{\Sigma_2^T y_{t-1}^2}. \tag{A18}
$$

Thus,

$$
Var(\hat{\rho}_{OLS}) = Var\left(\rho + \frac{\Sigma_2^T y_{t-1}\epsilon_t}{\Sigma_2^T y_{t-1}^2}\right) = Var(\rho) + Var\left(\frac{\Sigma_2^T y_{t-1}\epsilon_t}{\Sigma_2^T y_{t-1}^2}\right) + Cov\left(\rho, \frac{\Sigma_2^T y_{t-1}\epsilon_t}{\Sigma_2^T y_{t-1}^2}\right). \tag{A19}
$$

Since $Var(\rho) = 0$ and $Cov(\rho, \frac{\Sigma_2^T y_{t-1}\epsilon_t}{\Sigma_2^T y_{t-1}^2}) = 0$ because ρ is a fixed value. Therefore, we have

$$
Var(\hat{\rho}_{OLS}) = Var\left(\frac{\Sigma_2^T y_{t-1}\epsilon_t}{\Sigma_2^T y_{t-1}^2}\right). \tag{A20}
$$

The variance of functions of random variables to the second derivative is

$$
Var\left(\frac{X}{Y}\right) \approx \frac{Var(X)}{E(Y)^2} - \frac{2E(X)}{E(Y)^3}Cov(X,Y) + \frac{E(X)^2}{E(Y)^4}Var(Y). \tag{A21}
$$

From the last section, we know $E(\Sigma_2^T y_{t-1}\epsilon_t) = \Sigma_2^T E(y_{t-1}\epsilon_t) = 0$. Thus,

$$
\begin{aligned}
Var(\hat{\rho}_{OLS}) &\approx \frac{Var(\Sigma_2^T y_{t-1}\epsilon_t)}{E(\Sigma_2^T y_{t-1}^2)^2} \\
&= \frac{\sum_2^T Var(y_{t-1}\epsilon_t) + 2\sum\limits_{2\le t<\tau\le T} Cov(y_{t-1}\epsilon_t, y_{\tau-1}\epsilon_\tau)}{E(\sum_2^T y_{t-1}^2)^2}.
\end{aligned}
\tag{A22}
$$

According to equations A3 and A4,

$$\begin{aligned} Cov(y_{t-1}\epsilon_t, y_{\tau-1}\epsilon_\tau) &= E(y_{t-1}\epsilon_t y_{\tau-1}\epsilon_\tau) - E(y_{t-1}\epsilon_t)E(y_{\tau-1}\epsilon_\tau) \\ &= E(y_{t-1}\epsilon_t y_{\tau-1})E(\epsilon_\tau) - E(y_{t-1}\epsilon_t)E(y_{\tau-1}\epsilon_\tau) \\ &= 0 \end{aligned} \tag{A23}$$

when $t < \tau$.

Furthermore, we have (see equation 12 in the main text)

$$\begin{aligned} Var(\hat{\rho}_{OLS}) &\approx \frac{\Sigma_2^T Var(y_{t-1}\epsilon_t)}{E(\Sigma_2^T y_{t-1}^2)^2} \\ &= \frac{\Sigma_2^T (E(y_{t-1}^2)E(\epsilon_t^2) - E(y_{t-1})^2 E(\epsilon_t)^2)}{E(\Sigma_2^T y_{t-1}^2)^2} \\ &= \frac{\Sigma_2^T \frac{\sigma_\epsilon^2}{1-\rho^2}\sigma_\epsilon^2}{(\frac{(T-1)\sigma_\epsilon^2}{1-\rho^2})^2} \\ &= \frac{\frac{(T-1)\sigma_\epsilon^4}{1-\rho^2}}{\frac{(T-1)^2\sigma_\epsilon^4}{(1-\rho^2)^2}} \\ &= \frac{1-\rho^2}{T-1}. \end{aligned} \tag{A24}$$

This result is in accordance with the first-order asymptotic formula derived by Bartlett (1946).

A.3 The correlation between $\hat{\alpha}_{OLS}$ and $\hat{\rho}_{OLS}$ in the AR(1) model

Given that $\mu_{y_t} = \frac{\alpha}{1-\rho}$, the expected value of the OLS estimator of the intercept (see equation 13 in the main text) is

$$\begin{aligned} E(\hat{\alpha}_{OLS}) &= E\left(\frac{\sum_{t=2}^T y_t}{T-1} - \hat{\rho}_{OLS}\frac{\sum_{t=2}^T y_{t-1}}{T-1}\right) \\ &= E(y_t) - \hat{\rho}_{OLS}E(y_{t-1}) \\ &= (1-\hat{\rho}_{OLS})\mu_{y_t}. \end{aligned} \tag{A25}$$

Consequently, the bias of the OLS estimator of the intercept can be written as:

$$\begin{aligned} bias(\hat{\alpha}_{OLS}) &= E(\hat{\alpha}_{OLS}) - \alpha \\ &= (1-\hat{\rho}_{OLS})\mu_{y_t} - (1-\rho)\mu_{y_t} \\ &= (\rho - \hat{\rho}_{OLS})\mu_{y_t} \\ &= -bias(\hat{\rho}_{OLS})\mu_{y_t} \end{aligned} \tag{A26}$$

The correlation between $\hat{\alpha}_{OLS}$ and $\hat{\rho}_{OLS}$ is defined as $cor(\hat{\alpha}_{OLS}, \hat{\rho}_{OLS}) = \frac{cov(\hat{\alpha}_{OLS}, \hat{\rho}_{OLS})}{\sigma_{\hat{\alpha}_{OLS}}\sigma_{\hat{\rho}_{OLS}}}$, where $\sigma_{(\cdot)}$ represents the standard deviation. To find the variance and covariance of $\hat{\alpha}_{OLS}$ and $\hat{\rho}_{OLS}$, we know the variance-covariance matrix of the OLS estimators is $\sigma_\epsilon^2(X'X)^{-1}$, where X is the lagged value of Y (Hamilton, 1994). Taking $m_{kl} = \frac{1}{T}\sum_{t=1}^T y_{t-1}^k y_t^l$ for the integer powers k, l, $X'X$ could be represented as (Jarocinski and Marcet, 2010):

$$\begin{bmatrix} T & Tm_{10} \\ Tm_{10} & Tm_{20} \end{bmatrix}. \tag{A27}$$

We can derive $det(X'X) = T^2 m_{20} - T^2 m_{10}^2 = T^2 s_{y_{t-1}}^2$ where $s_{(\cdot)}^2$ is the sample variance. Thus, the variance-covariance matrix of the OLS estimators is

$$
\begin{aligned}
\Sigma_{\hat{\alpha}_{OLS},\hat{\rho}_{OLS}} &= \sigma_\epsilon^2 (X'X)^{-1} \\
&= \sigma_\epsilon^2 \begin{bmatrix} \frac{m_{20}}{Ts^2_{y_{t-1}}} & \frac{-m_{10}}{Ts^2_{y_{t-1}}} \\ \frac{-m_{10}}{Ts^2_{y_{t-1}}} & \frac{1}{Ts^2_{y_{t-1}}} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\sigma_\epsilon^2 m_{20}}{Ts^2_{y_{t-1}}} & \frac{-\sigma_\epsilon^2 m_{10}}{Ts^2_{y_{t-1}}} \\ \frac{-\sigma_\epsilon^2 m_{10}}{Ts^2_{y_{t-1}}} & \frac{\sigma_\epsilon^2}{Ts^2_{y_{t-1}}} \end{bmatrix}.
\end{aligned}
\tag{A28}
$$

Taking the variance and covariance of the estimators into the correlation formula brings us:

$$
\begin{aligned}
cor(\hat{\alpha}_{OLS}, \hat{\rho}_{OLS}) &= \frac{cov(\hat{\alpha}_{OLS}, \hat{\rho}_{OLS})}{\sigma_{\hat{\alpha}_{OLS}} \sigma_{\hat{\rho}_{OLS}}} \\
&= \frac{\frac{-\sigma_\epsilon^2 m_{10}}{Ts^2_{y_{t-1}}}}{\sqrt{\frac{\sigma_\epsilon^2 m_{20}}{Ts^2_{y_{t-1}}}} \sqrt{\frac{\sigma_\epsilon^2}{Ts^2_{y_{t-1}}}}} \\
&= \frac{-m_{10}}{\sqrt{m_{20}}}.
\end{aligned}
\tag{A29}
$$

A more elegant form of the correlation can be obtained by using a squared representation of equation A29:

$$
\begin{aligned}
cor(\hat{\alpha}_{OLS}, \hat{\rho}_{OLS})^2 &= \frac{m_{10}^2}{m_{20}} \\
&= \frac{m_{10}^2}{m_{10}^2 + s^2_{y_{t-1}}},
\end{aligned}
\tag{A30}
$$

where m_{10} is the sample mean of y_{t-1} and $s^2_{y_{t-1}}$ is the sample variance of y_{t-1}.

As the marginal distribution of y_t is known to be normally distributed, we can derive an asymptotic approximation of this formula (see equation 14 in the main text):

$$
cor(\alpha, \rho)^2 = \lim_{T \to \infty} cor(\hat{\alpha}_{OLS}, \hat{\rho}_{OLS})^2 = \frac{\alpha^2}{\sigma_\epsilon^2 \frac{1-\rho}{1+\rho} + \alpha^2}
\tag{A31}
$$

B supplementary material

B.1 Overview of correction methods

Although the two correction methods introduced in the paper are the primary focus, other correction methods have also been proposed and their performance investigated and published in the statistical and econometric literature. A comprehensive overview of the methodologies used, including characteristics, advantages, and disadvantages, is presented in Table 1.

In general, correction methods were classified into two principal categories: analytical functions and iterative, sampling, and simulation algorithms. These methods primarily reduce the level of small sample bias to $O(\frac{1}{T^2})$, which means that as T increases, the bias decreases at a rate proportional to $\frac{1}{T^2}$. We mainly considered analytical correction formulas because algorithmic methods are computationally intensive and do not outperform analytical methods (Engsted and Pedersen, 2014).

Table 1: Summary of methods for addressing the small sample bias in the AR effect

Correction Methods/Statistics	Reference	Comments
Bootstrap	Efron (1979)	Bias vanishes following the rate of the analytical formula derived by Engsted and Pedersen (2014).
$R = 2\rho - \frac{1}{2}(\rho_1 + \rho_2)^*$	Quenouille (1949)	A special case of bootstrap.
$E(\hat{\rho}_k) = \rho^k - \frac{2k\rho^k}{T} + O(\frac{1}{T^2})$ for AR(k) model**	Marriott and Pope (1954)	Kendall (1954) derived an analytical formula along with Marriott and Pope (1954).
Fixed point characterization***	Stine and Shaman (1989)	Stine and Shaman (1989) showed that the bias is moved toward the fixed point.
Median-unbiased estimator	Andrews (1993)	Does not hold when ρ is approximately 1.
$\hat{\rho'} = \hat{\rho} + fitting\,model$	Arnau Gras and Bono (2002)	Calculate by simulating the empirical bias of a conventional estimator
Bayesian correction with priors about initial conditions	Jarocinski and Marcet (2010)	Transform prior about initial conditions to prior of parameters

* ρ_1 and ρ_2 are the corresponding AR effects from the first and second halves of the series. This method reduce the bias to $O(\frac{1}{T^2})$.
** Yamamoto and Kunitomo (1984) also derived an analytical formula in VAR models, but Engsted and Pedersen (2014) has shown that it is identical to the one derived by Marriott and Pope (1954).
*** Bootstrap is an example of this technique, which involves two or more iterations.

B.2 Sensitivity analysis of the Bayesian correction

Figures 11 and 12 present the distribution of the AR estimates across 1000 replicates when varying the prior of α under different conditions of the AR(1) model and compare it with the spread of the OLS estimator and Kendall's correction. When $\alpha = 0$, the function between $\hat{\rho}$ and μ_α became weak when μ_α increases. However, in the other cases, a non-linear negative relation exists, and the Bayesian correction aligns with the true AR effect when $\mu_\alpha = \alpha$. Both figures in the sensitivity analysis are consistent with our discussion of equation A31.

Similarly, Figure 13 shows the distribution of the AR estimates across 1000 replicates when varying prior $\hat{\sigma}_\epsilon^2$ under different conditions of the AR(1) model and compares it with the spread of the OLS estimator and Kendall's correction. We notice that the color density of the AR estimates is constant for prior $\hat{\sigma}_\epsilon^2 \in [0.5, 20]$ and for different $\hat{\sigma}_\epsilon^2$ values when generating AR(1) time series. Thus, varying σ prior does not change the estimation of $\hat{\rho}$ corrected when $\hat{\sigma}_\epsilon^2$ is not set to a negligible value (i.e., $\hat{\sigma}_\epsilon^2 < 0.05$).

B.3 Figures

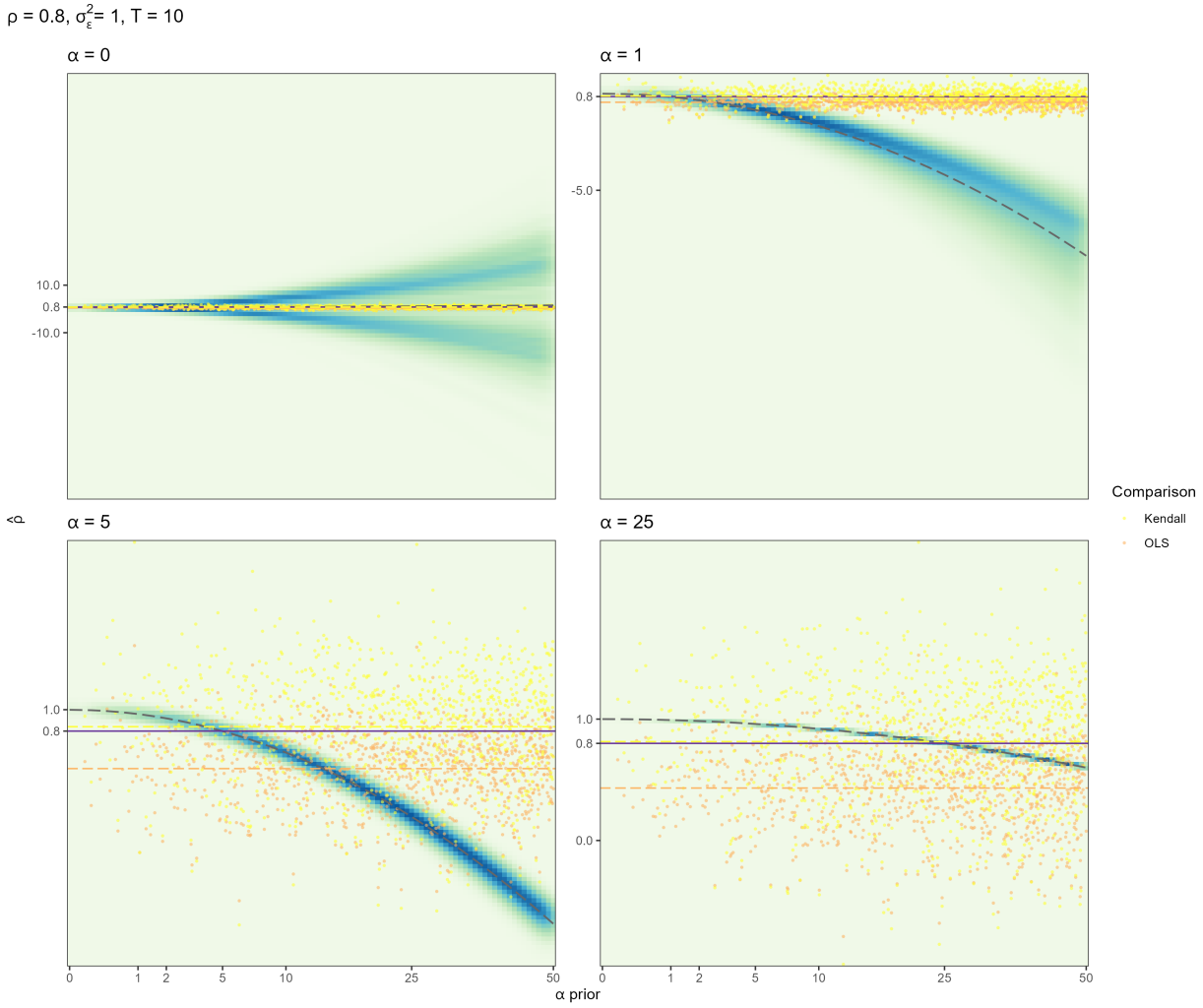

Figure 11: Density of estimated AR effects across 1000 replicates for prior $\mu_\alpha \in [0, 50]$. Dashed lines show the mean of Bayesian correction, Kendall's correction, and OLS estimation. The solid line represents the true AR effect.

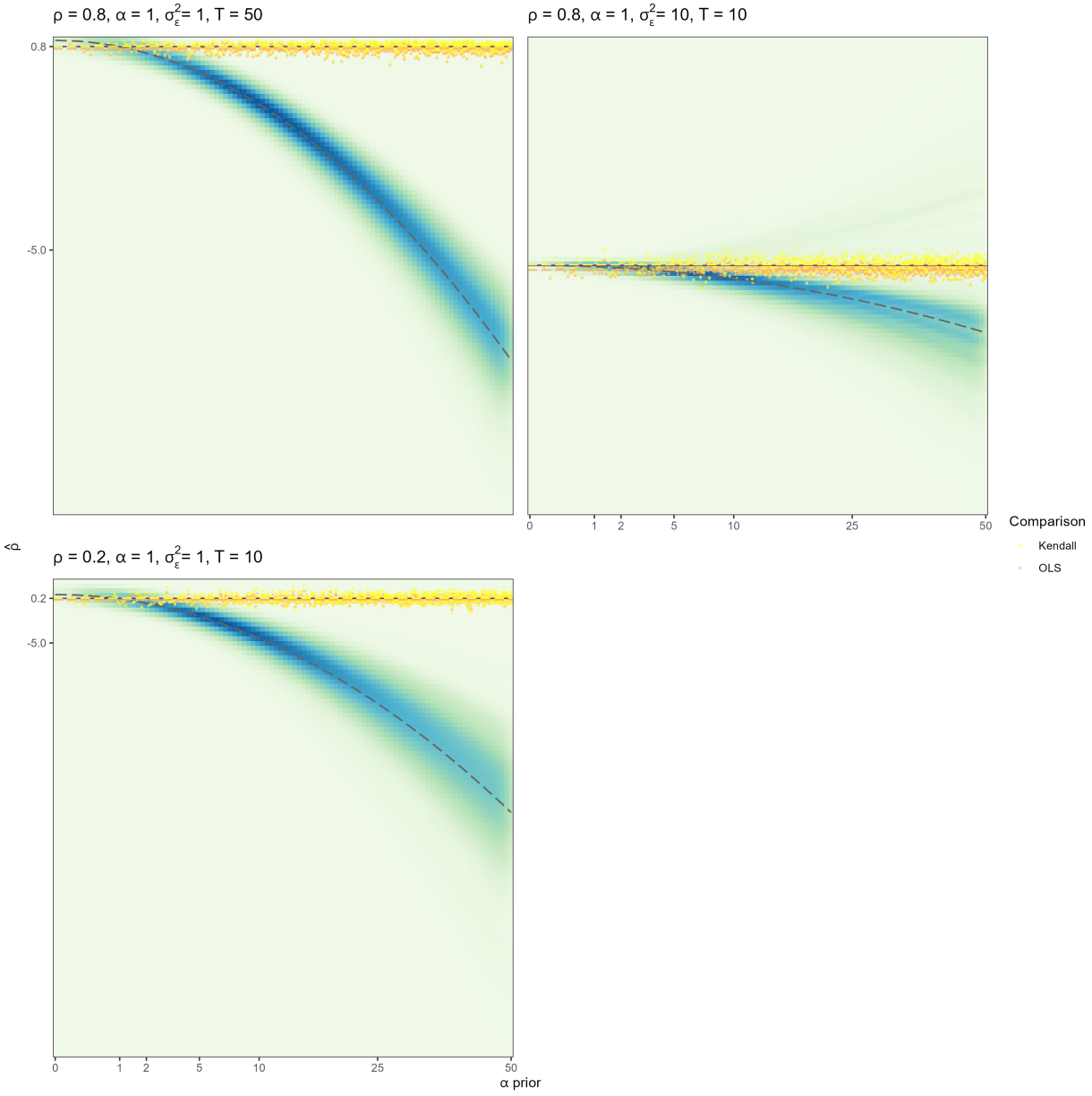

Figure 12: Density of estimated AR effects across 1000 replicates for prior $\mu_\alpha \in [0, 50]$. Dashed lines show the mean of Bayesian correction, Kendall's correction, and OLS estimation. The solid line represents the true AR effect.

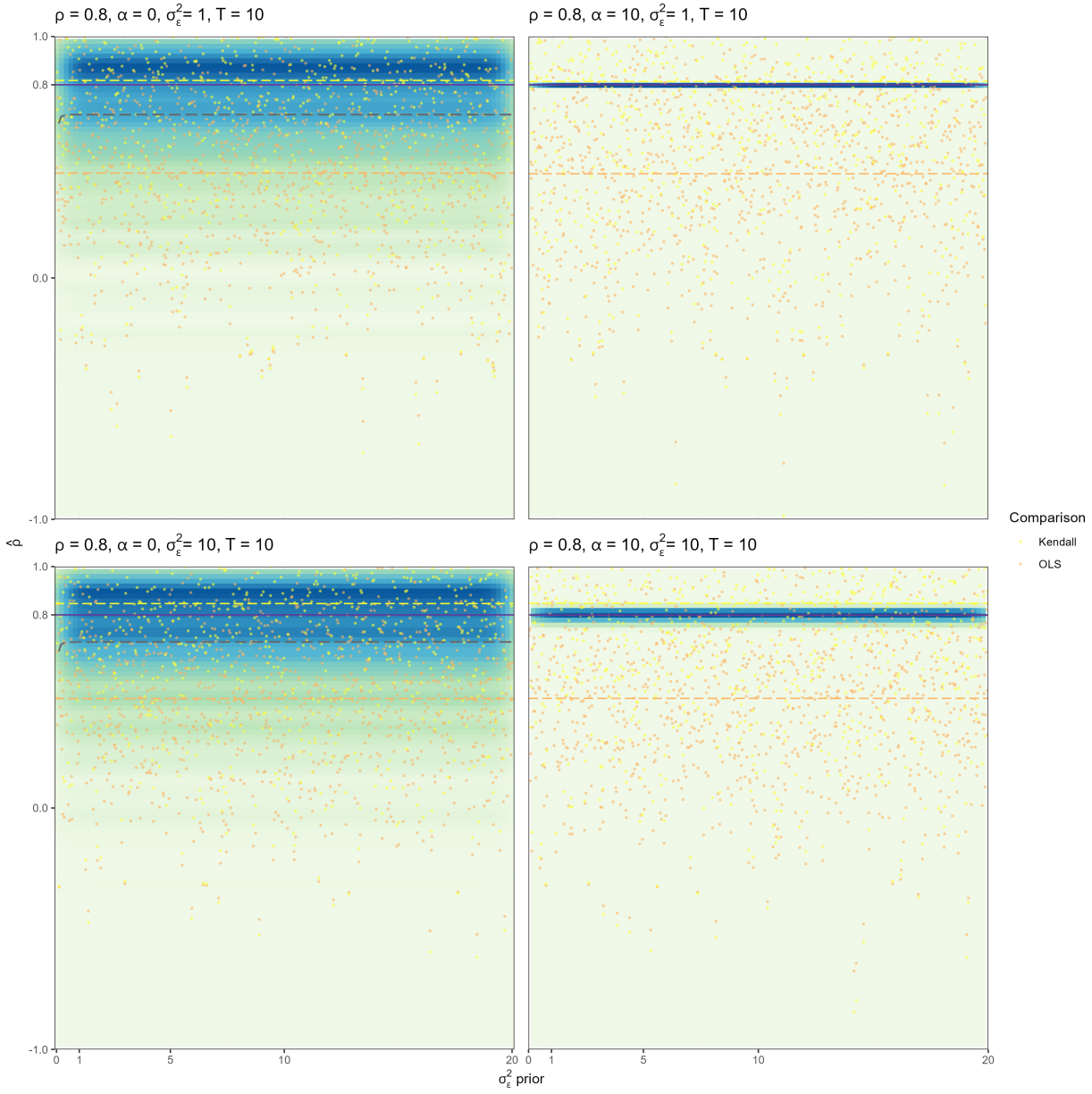

Figure 13: Density of estimated AR effects across 1000 replicates for prior $\sigma_\epsilon \in [0.1, 20]$. Dashed lines show the mean of Bayesian correction, Kendall's correction, and OLS estimation. The solid line represents the true AR effect.

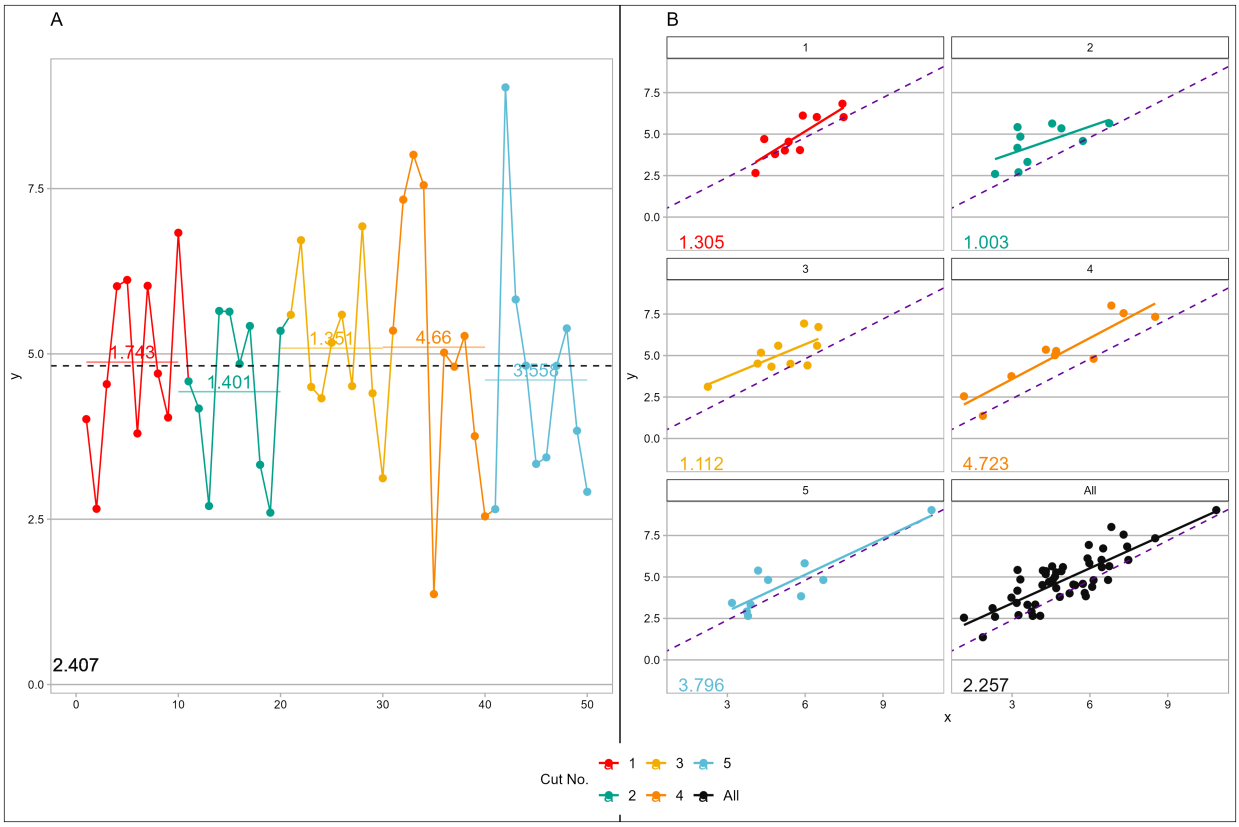

Figure 14: Data from a linear regression $y = 0.8x + \epsilon, x \sim N(0, \frac{25}{8}), \epsilon \sim N(0, 1)$ with 50 points. The 5 colors divide the data, including all observations, into 5 smaller datasets, each consisting of 10 observations. Panel A shows the observations of y. Solid lines and dashed lines represent the mean of each short dataset and the mean across all time points, respectively. Panel B displays the scatter plot of x versus y. The purple lines represent the true regression lines.

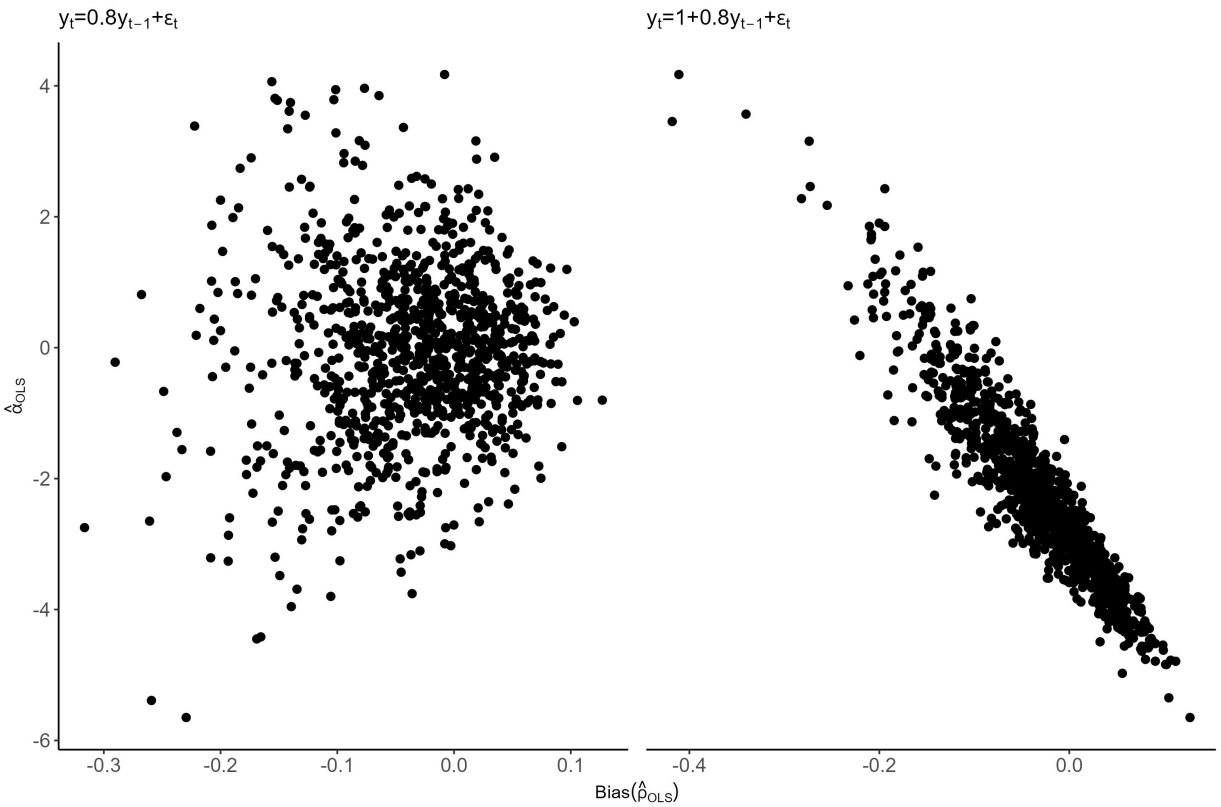

Figure 15: Bias of $\hat{\rho}_{OLS}$ versus $\hat{\alpha}_{OLS}$ across 1000 replicates, where $T = 100$, $\rho = 0.8$, $\sigma_\epsilon^2 = 1$, α equal to zero or one.

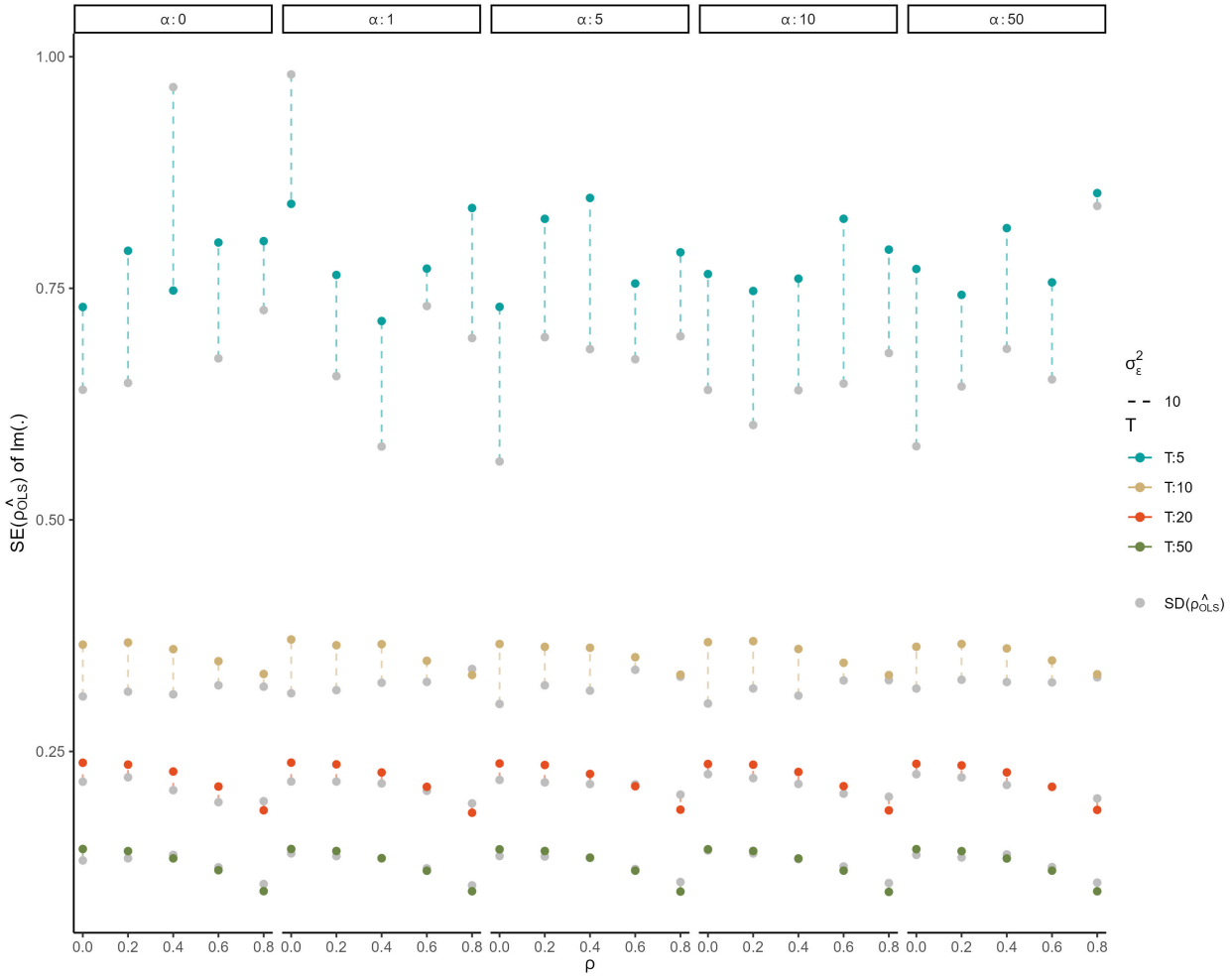

Figure 16: Simulation results: Comparison between standard error of OLS estimators from the function `lm(.)` and the true standard deviation of $\hat{\rho}_{OLS}$ when $\sigma_\epsilon^2 = 10$.

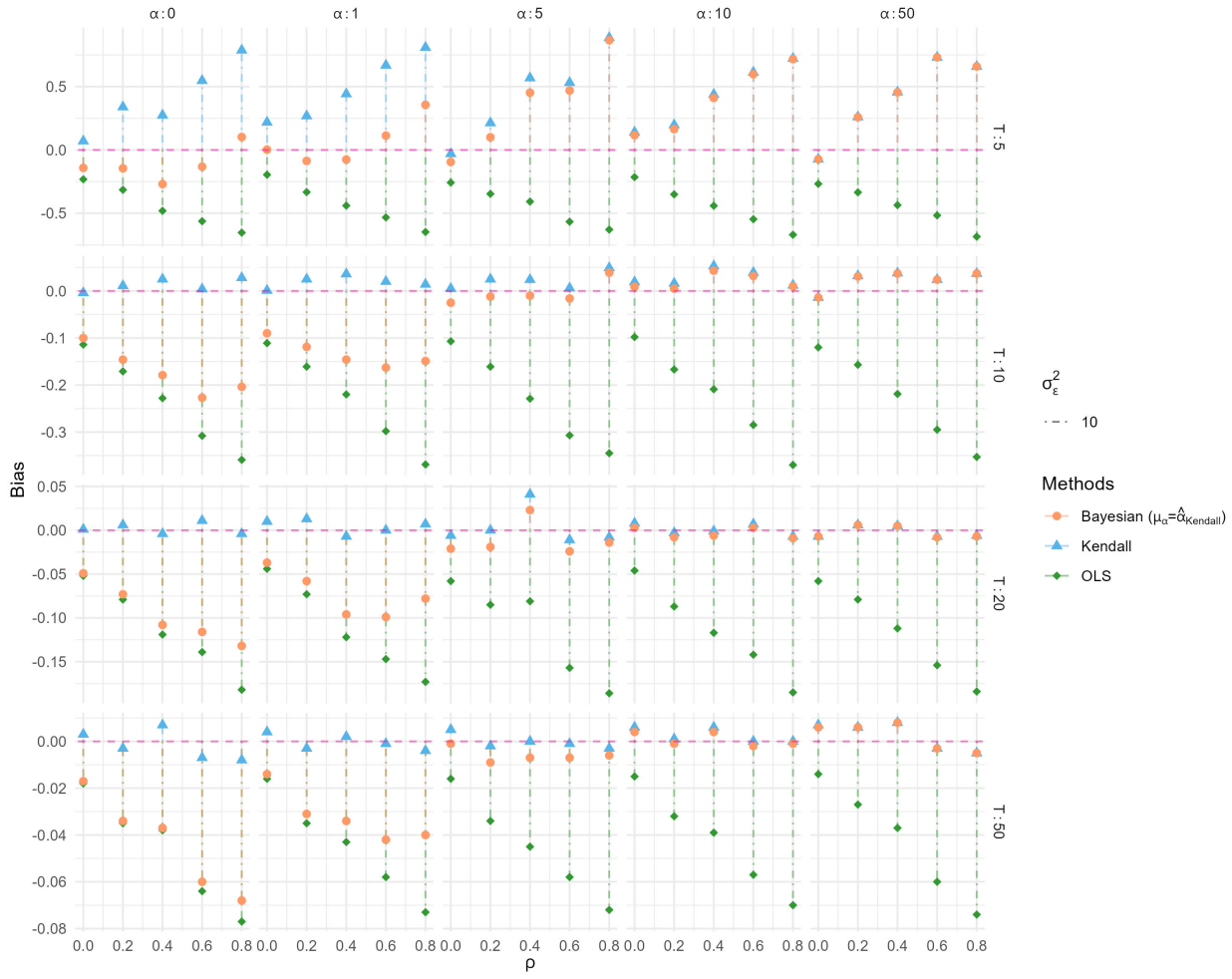

Figure 17: Simulation results: Bias of OLS estimators and corrected estimators when $\sigma^2_\epsilon = 10$.

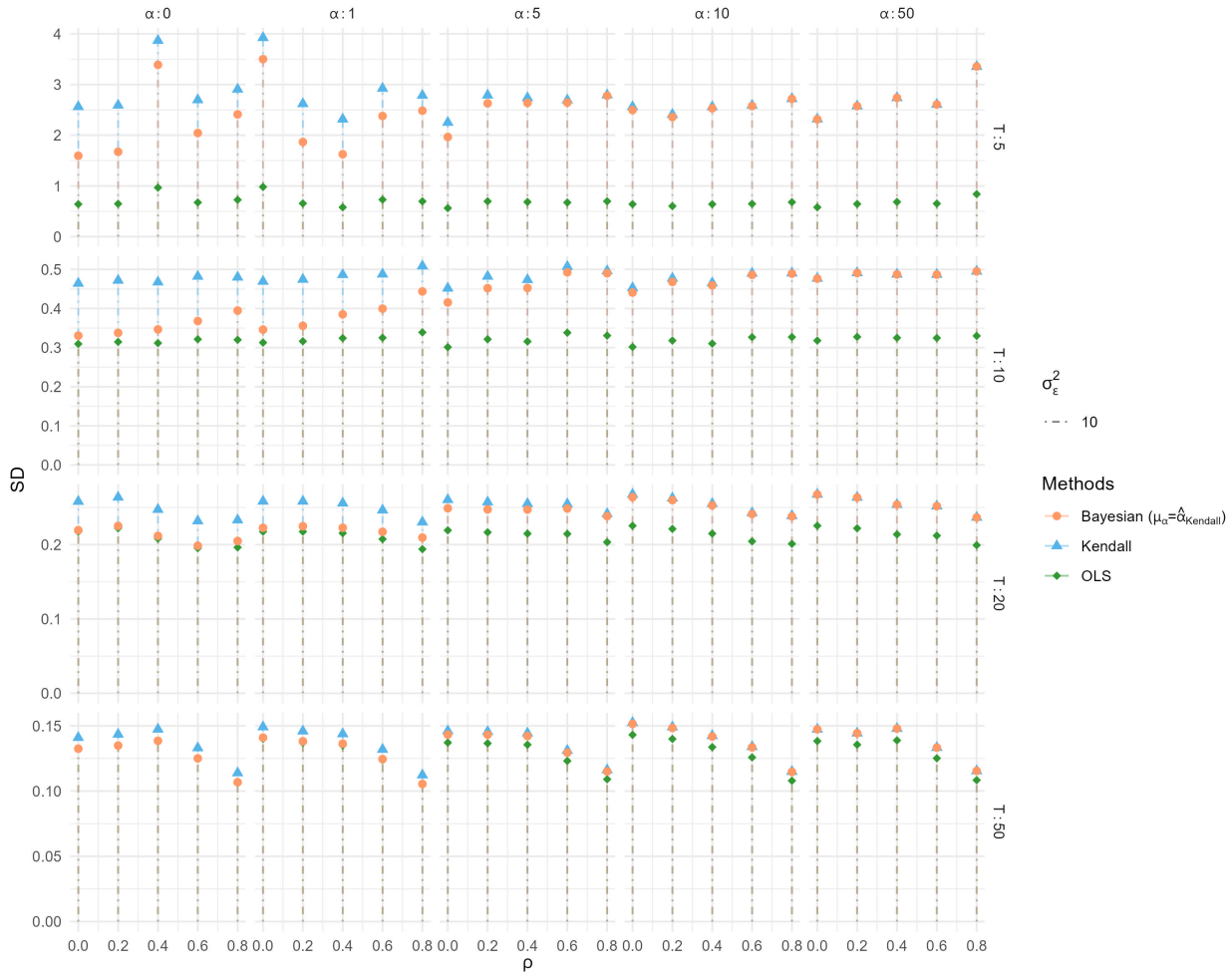

Figure 18: Simulation results: Mean ± SD of OLS estimators and corrected estimators when $\sigma^2_\epsilon = 10$. Mean is shown as the point, SD is shown as the radius of the range.

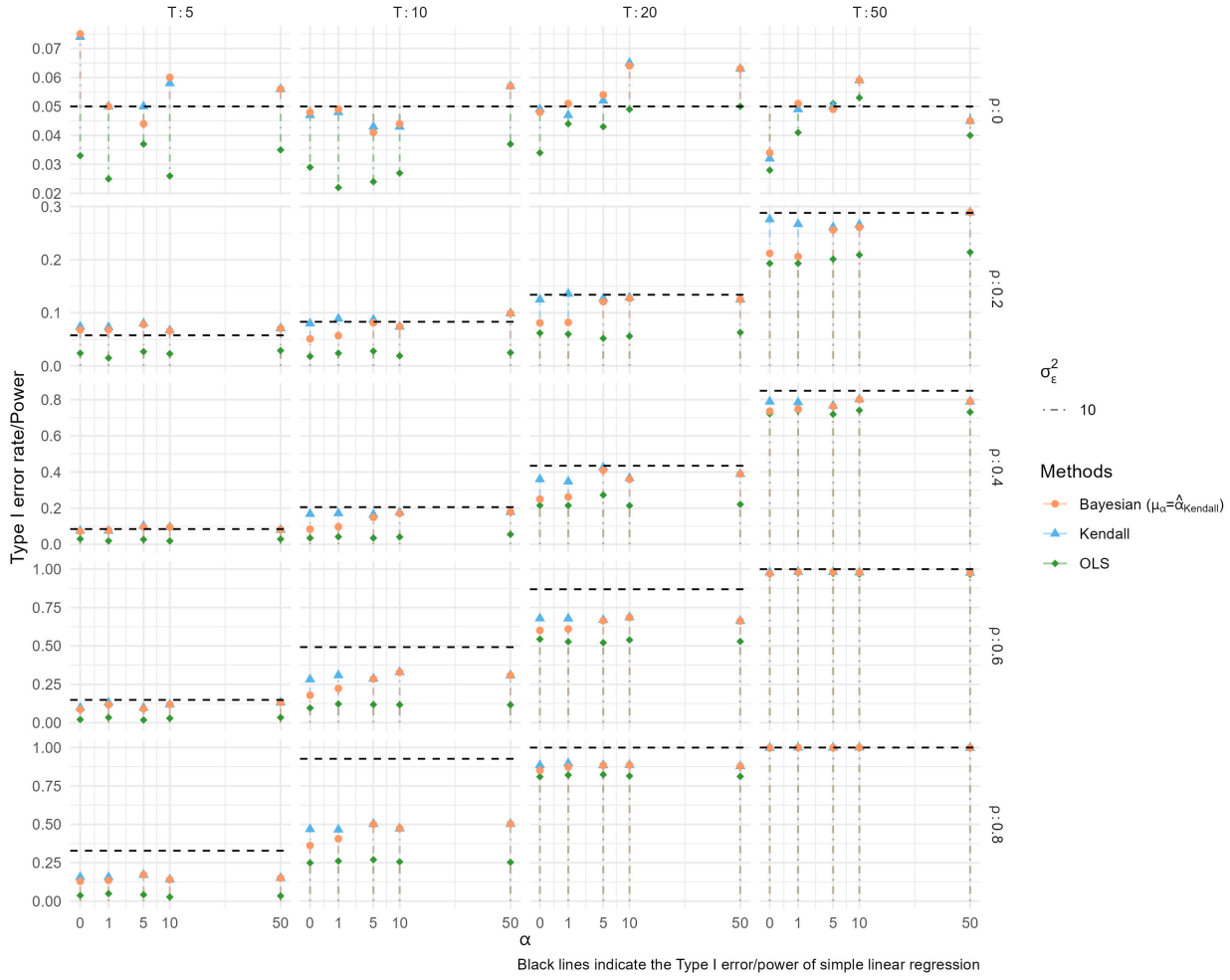

Figure 19: Simulation results: The first row of the sub figures presents the estimated Type I error rate and the second to fourth rows of the sub figures present the estimated power of OLS estimators and corrected estimators when $\sigma_\epsilon^2 = 10$. Pink lines display the nominal Type I error rate and statistical power of simple linear regression in each design cell.