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Collinearity Issues in Autoregressive Models with Time-Varying Serially Dependent Covariates

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ABSTRACT

First-order autoregressive models are popular to assess the temporal dynamics of a univariate process. Researchers often extend these models to include time-varying covariates, such as contextual factors, to investigate how they moderate processes' dynamics. We demonstrate that doing so has implications for how well one can estimate the autoregressive and covariate effects, as serial dependence in the variables can imply predictor collinearity. This is a noteworthy contribution, since in current practice serial dependence in a time-varying covariate is rarely considered important. We first recapitulate the role of predictor collinearity for estimation precision in an ordinary least squares context, by discussing how it affects estimator variances, covariances and correlations. We then derive a general formula detailing how predictor collinearity in first-order autoregressive models is impacted by serial dependence in the covariate. We provide a simulation study to illustrate the implications of the formula for different types of covariates. The simulation results highlight when the collinearity issue becomes severe enough to hamper interpretation of the effects. We also show that the effect estimates can be biased in small samples (i.e., 50 time points). Implications for study design, the use of time as a predictor, and related model variants are discussed.

KEYWORDS

Psychological dynamics; autoregressive models; collinearity; time series analysis

Introduction

Autoregressive (AR) models constitute one of the most widely used methods for modeling serial dependence in intensive longitudinal data (ILD). Such models regress the present state of a univariate process on states of the same process at preceding time points by including lagged versions of the measured variable as predictor variables (e.g. Lütkepohl, 2005). In recent years, these models have seen extensive use in many fields of psychological research that focus on within-person dynamics. Examples range from developmental psychology (e.g. Degoy & Olmos, 2020), to psychopathology research (Groen et al., 2019) or experimental fields such as the analysis of eye-tracking data (Cho et al., 2018). Models for psychological research that extend the key ideas of AR models are published frequently (e.g. Gistelinck & Loeys, 2020, Asparouhov et al., 2018, Beltz & Gates, 2017, Krone et al., 2018, Bulteel et al., 2018).

To further introduce the key ideas of AR models, we turn to the field of affect dynamics in which AR models are routinely applied to ILD of a person's affective states (e.g., E. Hamaker et al., 2015, Ariens et al., 2020), to investigate how emotional processes behave over time at the within-person level (e.g., Kuppens, 2015). The field of affect dynamics typically uses first-order AR (i.e. AR(1)) models, implying that a single AR effect between successive process states (i.e., each process state is only regressed on the previous one) is sufficient to capture the serial dependence. The basic AR(1) model has three parameters which are all of theoretical interest for the study of affect dynamics. The intercept plays the same role as an intercept in classical regression models. Statistically it expresses the expected process state when all predictors equal 0. In AR(1) models it is often included to allow for non-zero means of affective processes, establishing the parameter's theoretical contribution (Bringmann et al., 2017)[1]. The residual variance, also called the *innovation variance*, is an important component of the variability of the emotional process. The emotional variability also depends on the party piece

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[1]Note however that the intercept is not the same as the mean, unless one mean-centers the ILD before the analysis

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of the AR(1) model, which is that it allows process states at time t to depend on the previous process state at time $t-1$, by means of the regression weight, or AR effect, of the lagged predictor. This AR effect is theoretically interpreted as an index of the extent to which an emotional process displays *emotional inertia* (Houben et al., 2015, Kuppens et al., 2010).

From the above, it is clear that psychologists tend to assign a close correspondence between the statistical parameters of AR(1) models and theoretical constructs. The powerful combination of these perceived relationships and the simplicity of the basic AR(1) model have spurred its widespread use. Whether there are inter-individual differences in the AR effect and whether these are meaningfully related to relevant person-level outcomes, has been a topic of considerable interest in recent publications (Koval et al., 2013, Bosley et al., 2019). However, its application comes with a number of assumptions which can be rather restrictive when applied in practice, such as the assumption that the process on which an AR(1) model is fitted displays a form of stability over time known as stationarity. This means that the process should display constant mean, variance, and autocorrelations over time. However, many substantive questions, (implicitly) assume that there is some form of change in affective dynamics over time (Bringmann et al., 2018, Adolf et al., 2017). The interest in the onset and development of pathological emotional processing (e.g. Lamers et al., 2018), an increase in average self-reported stress over time (e.g. Wang et al., 2020), a sudden change in emotional variability or stability following a major life event (e.g. Betz et al., 2020, Simons et al., 2021), or differences in the extent to which an emotional process displays inertia for instance (e.g. Waugh et al., 2017, Koval & Kuppens, 2012), can not be accommodated by the basic AR(1) model (see also Bringmann et al., 2017).

A popular route to model such changes in dynamics is the inclusion of relevant contextual information which may explain them, an approach which will be the topic of this paper. This can be achieved by appending observed time-varying covariates (e.g. different experimental stimuli or the occurrence of positive or negative events) to the basic AR(1) model (e.g. Lamers et al., 2018, Basinska & Gruszczynska, 2020 Kuppens, 2015, Brans et al., 2013). These covariates may excert their influence as main effects, altering the intercept and consequently the mean of the process at each time point. Next to including information on events or stimuli, adding time (or polynomials of time) as a separate predictor variable is often done to control for changes in mean over time (e.g. Jebb et al., 2015). Besides allowing for such intercept changes many researchers are interested in assessing changes in the AR effect (e.g. Koval & Kuppens, 2012). To this end, they include product terms of the lagged predictor and the covariate. Some researchers have also considered modeling changes in innovation variance (Adolf et al., 2017). It is important to realize that contextual predictors can be included in a lagged or contemporaneous way. In the lagged case, only the previous value of the covariate affects the process at time t, reflecting the idea that effects of the covariate unfold over time (Adolf et al., 2017). Contemporaneous effects imply that the influence of the covariate on the process is immediate. This might be more appropriate if the time intervals between measurements are large, while the effects of the covariate are rather short-lived (Schuurman et al., 2015), or almost immediate after their occurrence, for instance.

Although very useful from a substantive point of view, little attention has been paid to how including such time-varying covariates impacts the accuracy with which the parameters of the AR(1) model can be estimated. While omitting covariates can of course lead to biased estimates (e.g., of the AR effect), it is well known from the standard regression literature (e.g. Cohen et al., 2013) that estimation precision may be jeopardized in case of substantial collinearity between the predictors in the estimated model. Here, we use estimation precision to refer to estimator variances, and the related standard errors and confidence intervals, but also to estimator covariances and correlations, resulting in 'tradeoff' behavior between estimators (cf. Leamer, 1973).

In this paper, we will show analytically and by means of simulations that collinearity is more of an issue in AR(1) models with time-varying covariates, as it can be amplified by the time series-nature of the variables in the model and their serial dependence. Specifically, using ordinary least squares (OLS) estimation theory, we first recapitulate in the section "Estimation & collinearity of predictors" how the variances, covariances and correlations of regression coefficient estimators depend on the covariance and correlation (and hence collinearity) between predictors.

In the section "Collinearity in AR(1) models with a single time-varying covariate", we then analytically derive the predictor covariance as implied by the here-considered model (i.e., the covariance between the covariate and the lagged outcome), and show *that*

it is affected by serial dependence in the covariate. This result holds regardless of the nature of the covariate. For the special case in which the covariate itself follows an AR(1) process, we can further show *how* serial dependence in the covariate, specifically its AR effect, affects the predictor covariance and correlation in interplay with the other model parameters. This establishes a direct link between (standardized) serial dependence in the covariate and predictor collinearity.

In the subsequent section, we complement these results with a comprehensive simulation study, in which we verify and illustrate our analytical results examining estimation accuracy of AR(1) models with covariates of different formats and shapes and for plausible parameter effect sizes. We also show that in small samples, estimation precision can drop substantially, especially with highly autocorrelated predictors such as time trends. Furthermore, small-sample biases for the autoregressive effects can affect the other estimators in the case of high estimator correlations. In severe cases, the problem can be extensive enough to distort theoretical interpretation of the coefficients. In the discussion, we reflect on the implications of these findings for study design and for extensions of the model that we studied.

Autoregressive models and their estimation

The basic AR(1) model

The basic AR(1) model assumes that the values of the process at each time point, y_t, with $t = 1, ..., T$ indexing the T measurement occasions, can be predicted based on the values of the preceding observation, y_{t-1}. This model implies that only the most recent past is informative about the state of the process at time t. In equations, an AR(1) model is given by

$$y_t = \beta_0 + \beta_1 y_{t-1} + \varepsilon_t, \text{with } \varepsilon_t \sim N(0, \sigma_\varepsilon^2), \quad (1)$$

where β_0 is the intercept, β_1 is the AR effect, and ε_t is the error term capturing the so-called process innovations over time. As their name suggests, the innovations are assumed to be independent of past process states up to and including y_{t-1}, serially independent, and distributed according to a zero-mean normal distribution with constant variance[2], σ_ε^2. In addition, the AR effect must be smaller than 1 in absolute value for an AR(1) model to display *covariance stationarity*. In that case, the process displays a constant mean, which is given by $\mu_y = \frac{\beta_0}{(1-\beta_1)}$, constant variance, given by $\sigma_y^2 = \frac{\sigma_\varepsilon^2}{(1-\beta_1)^2}$, and a constant autocorrelation β_1 (Hamilton, 1994). The innovations can then be thought of as perturbations, which drive the process away from its mean and feed forward through time due to the presence of the AR effect. Specifically, the size of the AR effect indicates how far perturbations feed forward and how quickly the process returns to its mean. The closer the AR coefficient is to 1, the longer it will take for the effect of an innovation to" die out", and the slower the process will revert to its mean (Hamilton, 1994, p. 54).

The AR(1) model with a single time-varying covariate

As mentioned in the introduction, time-varying covariates are often included in an AR(1) model in order to assess whether and how the model parameters depend on the contextual information reflected in the covariate scores. These effects can be conceived as contemporaneous or as lagged. We limit the following presentation to the case of a single covariate for reasons of simplicity. Specifically, to investigate how the intercept changes as a function of the covariate scores at the preceding observation (i.e., a lagged effect), one can include the main effect of the lagged time-varying covariate:

$$y_t = \beta_0 + \beta_1 y_{t-1} + \beta_2 x_{t-1} + \varepsilon_t. \quad (2)$$

That this implies a change in the intercept becomes evident by rewriting the equation as

$$y_t = \beta_{0t}^* + \beta_1 y_{t-1} + \varepsilon_t, \text{with } \beta_{0t}^* = \beta_0 + \beta_2 x_{t-1}. \quad (3)$$

The contemporaneous effect of the covariate can be studied by replacing x_{t-1} by x_t in these equations. While the innovation assumptions mentioned for the AR(1) model without covariates remain, the innovations at time t should also be independent of the covariate(s) at time $t-1$ (for lagged effects) or time t (for contemporaneous effects). This - as well as y_{t-1} being independent of ε_t - is termed contemporanous exogeneity of the predictors and guarantees *asymptotic*[3] unbiasedness when estimating their effects (Wooldridge, 2012, p.349).

Including an interaction effect between the lagged covariate and the lagged predictor clears the way to additionally study how the AR effect changes in function of the covariate scores:

[2]implying homoskedasticity (Wooldridge, 2012, p. 243)

[3]i.e. as the sample size tends to infinity

$$\begin{aligned} y_t &= \beta_0 + \beta_1 y_{t-1} + \beta_2 x_{t-1} + \beta_3 x_{t-1} y_{t-1} + \varepsilon_t \\ &= (\beta_0 + \beta_2 x_{t-1}) + (\beta_1 + \beta_3 x_{t-1}) y_{t-1} + \varepsilon_t \\ &= \beta_{0t}^* + \beta_{1t}^* y_{t-1} + \varepsilon_t, \text{with } \beta_{0t}^* = \beta_0 + \beta_2 x_{t-1} \end{aligned}$$

and $\beta_{1t}^* = \beta_1 + \beta_3 x_{t-1}$. (4)

In this paper, we will mostly focus on the model where only the intercept changes, but come back to the model with a changing AR effect in the discussion.

Estimation and collinearity of predictors

So far, we have considered the AR models of interest at the level of the population. However, the true population-level parameters, which are denoted by greek letters, e.g., β_1, are typically unknown and therefore estimated from data. We discuss the case of OLS estimation in the following. To distinguish parameter estimators or estimates from population-level parameters we use the hat-notation, e.g., $\hat{\beta}_1$.

As indicated in the introduction, we can estimate the parameters of AR(1) models with or without time-varying covariates by means of OLS estimation, based on the assumptions about the innovations that were mentioned higher (see also Wooldridge, 2012). Note that the issue at hand (i.e., potentially impaired estimation precision through predictor collinearity) is not specific to OLS estimators[4], since the origin of the collinearity lies in the data-generating structure, not in how the model is estimated. To recapitulate how OLS estimation works, we rewrite the models in matrix form:

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}. \quad (5)$$

Hereby, $\boldsymbol{y}$ and $\boldsymbol{\varepsilon}$ are $T \times 1$ vectors containing the T scores y_t and innovations respectively, $\boldsymbol{X}$ is the $T \times (P+1)$ *design matrix*, containing a first column of 1's for the intercept and P additional columns for the P predictors and $\boldsymbol{\beta}$ is a $(P+1) \times 1$ vector containing the intercept and the regression weights. For the AR(1) model without covariates, the design matrix thus has two columns, the column of 1's and a column containing the lagged outcome values y_{t-1}. When including a covariate to model changes in intercept, we append an additional column to the design matrix, containing the lagged x_{t-1} or contemporaneous x_t values. The OLS estimators for the regression coefficients, $\hat{\boldsymbol{\beta}}$, are then obtained as:

$$\hat{\boldsymbol{\beta}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y}. \quad (6)$$

The asymptotic sampling distribution of the OLS estimators is a multivariate normal distribution with mean vector $\boldsymbol{\mu}_{\hat{\boldsymbol{\beta}}} = \boldsymbol{\beta}$ and covariance matrix (Hamilton, 1994, p. 201 f.):

$$\boldsymbol{\Sigma}_{\hat{\boldsymbol{\beta}}} = \sigma_\varepsilon^2 (\boldsymbol{X}'\boldsymbol{X})^{-1}. \quad (7)$$

The covariance matrix of this sampling distribution informs us about the variances of each of the parameter estimators and their covariances, thus shedding light on estimation precision. The larger the variances, the less precise the obtained estimates are, and the wider the associated confidence intervals will be. In turn, large negative covariances (if they translate to large negative correlations) imply that parameter estimates trade off for each other.

We therefore take a closer look at the OLS estimators' covariance matrix. We first note that it is proportional (by a factor of σ_ε^2) to the inverted matrix $(\boldsymbol{X}'\boldsymbol{X})^{-1}$. The matrix $\boldsymbol{X}'\boldsymbol{X}$ in turn is proportional (by a factor of T) to the covariance matrix of the predictors, $\boldsymbol{S_x}$ (including the column of 1's), plus the products of their means with $\boldsymbol{m_x}$ being a $(P+1) \times 1$ vector of predictor means (Ledoit & Wolf, 2003):

$$\boldsymbol{X}'\boldsymbol{X} = T\left(\boldsymbol{S_x} + \boldsymbol{m_x}\boldsymbol{m_x}'\right), \quad (8)$$

From this we can already see that the covariance of the lagged outcome and the lagged or contemporaneous covariate, which is an entry of matrix $\boldsymbol{S_x}$ if the covariate is among the predictors, matters for estimation precision. This becomes explicit by deriving the estimator covariance matrix for the AR(1) model with a lagged covariate element-wise, shedding light on the composition of its individual elements, the variances and covariances among the estimators $\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2$ in $\hat{\boldsymbol{\beta}}$ (see Appendix A for more detail):

$$\boldsymbol{\Sigma}_{\hat{\boldsymbol{\beta}}} = \begin{bmatrix} \sigma^2_{\hat{\beta}_0} & \sigma_{\hat{\beta}_0\hat{\beta}_1} = \dfrac{\sigma_\varepsilon}{T\left(1 - r^2_{y_{t-1}x_{t-1}}\right)s^2_{y_{t-1}}} & \sigma_{\hat{\beta}_0\hat{\beta}_2} = -\dfrac{r_{y_{t-1}x_{t-1}}\sigma_\varepsilon}{T\left(1 - r^2_{y_{t-1}x_{t-1}}\right)s_{y_{t-1}}s_{x_{t-1}}} \\ \sigma_{\hat{\beta}_0\hat{\beta}_1} & \sigma^2_{\hat{\beta}_1} = \dfrac{\sigma_\varepsilon}{T\left(1 - r^2_{y_{t-1}x_{t-1}}\right)s^2_{y_{t-1}}} & \sigma_{\hat{\beta}_1\hat{\beta}_2} = -\dfrac{r_{y_{t-1}x_{t-1}}\sigma_\varepsilon}{T\left(1 - r^2_{y_{t-1}x_{t-1}}\right)s_{y_{t-1}}s_{x_{t-1}}} \\ \sigma_{\hat{\beta}_0\hat{\beta}_2} & \sigma_{\hat{\beta}_1\hat{\beta}_2} = -\dfrac{r_{y_{t-1}x_{t-1}}\sigma_\varepsilon}{T\left(1 - r^2_{y_{t-1}x_{t-1}}\right)s_{y_{t-1}}s_{x_{t-1}}} & \sigma^2_{\hat{\beta}_2} = \dfrac{\sigma_\varepsilon}{T\left(1 - r^2_{y_{t-1}x_{t-1}}\right)s^2_{x_{t-1}}}, \end{bmatrix} \quad (9)$$

Hereby, $s_{x_{t-1}}$ and $s_{y_{t-1}}$ denote the standard deviations of the covariate x_{t-1} and lagged outcome y_{t-1} and $r_{y_{t-1}x_{t-1}}$ denotes their correlation. We can in addition derive that the correlation between the regression weight estimators $\hat{\beta}_1$ and $\hat{\beta}_2$ equals the predictor correlation in absolute value, but has the opposite sign:

$$r_{\hat{\beta}_1\hat{\beta}_2} = \frac{\sigma_{\hat{\beta}_1\hat{\beta}_2}}{\sqrt{\sigma^2_{\hat{\beta}_1}}\sqrt{\sigma^2_{\hat{\beta}_2}}} = -r_{y_{t-1}x_{t-1}} \quad (10)$$

[4]For the models considered, OLS and maximum likelihood estimation will lead to the same results asymptotically (Lütkepohl, 2005). We verified (but do not report here) that applying maximum likelihood estimation to our simulated data in section 4 leads to highly similar results.

Thus, if the correlation between the predictors $r_{y_{t-1}x_{t-1}}$ tends to 1 (i.e., the predictors become highly collinear), the correlation of $\hat{\beta}_1$ and $\hat{\beta}_2$ will tend to -1 and the variances of these estimators will tend to infinity [5]. This points to the crucial role of the predictor correlations in estimation precision. Severe collinearity can thus hamper the interpretation of regression weights (e.g. see Cohen et al., 2013, p. 419). Under high collinearity, slight changes in the data can lead to disproportionate changes in the estimated coefficients (a phenomenon known as bouncing beta behavior (Smith & Sasaki, 1979)). The tradeoff behavior between correlated estimates further renders their interpretation as unique direct effects problematic.

Obviously, low predictor correlations are not worrisome, raising the question when predictor correlations are strong enough to be considered indicative of a collinearity problem. Although collinearity naturally is a gradual characteristic, heuristics vary and range from bivariate pedictor correlations exceeding .7 (e.g. Dormann et al., 2013) or .8 (e.g. Slinker & Glantz, 1985, Willis & Perlack, 1978). In this paper we consider any predictor correlation achieving such values as potentially troublesome. Specifically, we will show that the predictor correlations in AR(1) models with time-varying covariates can often reach such levels due to the serial dependence of the variables involved.

Collinearity in AR(1) models with a single time-varying covariate

This section presents our analytic results for predictor collinearity in AR(1) models with a single lagged or contemporaneous time-varying covariate. Studying the occurrence of predictor collinearity in this setting, where researchers endow the regression weights with theoretical interpretations, is crucial, because it renders this mapping between theory and statistical parameters shaky. In a first subsection, we present model-implied expressions for the predictor covariance $s_{y_{t-1}x_{t-1}}$ under the model with a lagged and a contemporaneous covariate. In deriving these (see appendix for details), we do not make assumptions regarding how the covariate values were generated. The expressions are thus *general* in the sense that they should accommodate any time-varying covariate. Of course, the presence of predictor covariance alone only tells us that there is collinearity, but not to which degree. Unfortunately, the generality comes at the price that an accessible expression of the predictor correlation cannot straightforwardly be derived. In a second subsection, we therefore turn to the specific case where the time-varying covariate itself follows an AR(1) process. For this case we again show expressions for the model-implied predictor covariance, but also provide one for the predictor correlation in Appendix B.2. In the main text, we visualize the behavior of the predictor covariance and correlation as a function of serial dependence in the covariate, in this case its AR effect, among the other model parameters. We close by briefly discussing the implications of our analytic findings.

General expressions

For the lagged covariate case, we find that the covariance of the predictors $s_{y_{t-1}x_{t-1}}$ can be generally (i.e., for any time-varying covariate) decomposed as (see Appendix B.1):

$$s_{y_{t-1}x_{t-1}} = \beta_1 s_{y_{t-2}x_{t-1}} + \beta_2 s_{x_{t-2}x_{t-1}} + s_{\varepsilon_{t-1}x_{t-1}}. \quad (11)$$

This expression still features a covariance term concerning both predictors on the right-hand side (i.e., $s_{y_{t-2}x_{t-1}}$). Assuming that our AR(1) process with the lagged covariate effect started a long time ago, and we have obtained sufficiently many observations, we can further decompose this term by recursive substitution. Going K steps back in time, we get:

$$s_{y_{t-1}x_{t-1}} = \beta_1^K s_{y_{(t-1)-K}x_{t-1}} + \sum_{k=1}^{K} \beta_1^{k-1}\left(\beta_2 s_{x_{(t-1)-k}x_{t-1}} + s_{\varepsilon_{t-k}x_{t-1}}\right). \quad (12)$$

If we take it to the extreme and let K approach ∞, then β_1^K approaches zero since $|\beta_1| < 1$ (as a necessary condition for stationarity, see section "The basic AR(1) model"). As a consequence, the term involving the covariance $s_{y_{(t-1)-K}x_{t-1}}$ becomes negligibly small and the expression reduces to:

$$s_{y_{t-1}x_{t-1}} = \sum_{k=1}^{\infty} \beta_1^{k-1}\left(\beta_2 s_{x_{(t-1)-k}x_{t-1}} + s_{\varepsilon_{t-k}x_{t-1}}\right). \quad (13)$$

Equation 13 highlights the impact of the autocovariance(s) of the covariate on the predictor covariance. This obviously also holds for the case where the covariate excerts its influence contemporaneously, in which case we obtain (see Appendix B.1):

$$s_{y_{t-1}x_t} = \beta_1 s_{y_{t-2}x_t} + \beta_2 s_{x_{t-1}x_t} + s_{\varepsilon_{t-1}x_t}, \quad (14)$$

and subsequently

[5] In case of *perfect* collinearity, the model is no longer identifiable since the predictors no longer convey any unique information, and their relative effects can no longer be disentangled (Cohen et al., 2013, p.419)

$$s_{y_{t-1}x_t} = \sum_{k=1}^{\infty} \beta_1^{k-1}(\beta_2 s_{x_{t-k}x_t} + s_{\varepsilon_{t-k}x_t}). \quad (15)$$

Expressions when the covariate follows an AR(1) process

To get more grip on these expressions and the role of the covariate's serial dependence, we studied how the expressions can be rewritten in case the covariate itself is also generated by an AR(1) process:

$$x_t = \gamma x_{t-1} + \zeta_t, \quad (16)$$

with $|\gamma| < 1$.

We now distinguish two cases for both lagged and contemporaneous effects: a more general one in which the innovations of x_t, ζ_t, are contemporaneously related to the innovations of y_t, ε_t, and a more restricted one in which they are unrelated. In both cases ζ_t and ε_t are jointly normally distributed with a zero mean vector and a covariance matrix that is full in the first case and diagonal in the second case. The variance of ζ_t is denoted by σ_ζ^2, the innovations' covariance by $\sigma_{\zeta\varepsilon}$.[6]

In the more general case of related innovations, the expression for the predictor covariance for the model with the lagged covariate presented in Equation 13 can be rewritten into (see Appendix B.2):

$$s_{x_{t-1}y_{t-1}} = \frac{\beta_2\,\gamma\,\frac{\sigma_\zeta^2}{1-\gamma^2}}{1-(\beta_1\,\gamma)} + \frac{\sigma_{\zeta\varepsilon}}{1-(\beta_1\,\gamma)}. \quad (17)$$

Next to β_1 and β_2, the first term of this expression clearly depends on the AR effect of the covariate, γ, and hence its autocorrelation. The second term contains the covariance of the innovations. In the more restricted case where the innovations are assumed to be uncorrelated (i.e., $\sigma_{\zeta\varepsilon} = 0$) this second term obviously drops out. Note that this expression also holds for the model-implied predictor covariance under the model with the contemporaneous covariate (i.e., Equation 15; for more detail see Appendix B.2). The expression of the model-implied predictor correlation under both covariate cases can be found in Appendix B.2.

We can use the expression of the predictor covariance (17) and the expression of the predictor correlation (B.48) to better understand how both depend on the model parameters β_1, β_2 and γ. We visualize this dependence for the predictor covariance in panel A of Figure 1, assuming uncorrelated innovations, an innovation variance σ_ε^2 of 1, and a covariate variance σ_x^2 of 1. In addition, we also display the corresponding implied predictor correlations in panel B. Figure 1 clearly shows that as γ increases, so do the predictor covariances and correlations, in all cases where $\beta_2 \neq 0$. In addition, if $\gamma = 0$ then the predictor covariance remains 0 regardless of β_1 or β_2. Finally, increasing β_1 mainly changes the shape of the functions linking the predictor covariances and correlations to γ, from more linear shapes to more curved ones, leading to increased covariances and correlations for larger sizes of γ. Importantly, multiple combinations of β_1, β_2 and γ lead to large predictor correlations (e.g. $r_{x_{t-1}y_{t-1}} > .7$), pointing toward potential collinearity problems.

Implications

We conclude that serial dependence in a covariate that has a non-zero effect ($\beta_2 \neq 0$) on the outcome, will induce (additional) covariance and correlation between the predictors of the AR(1) model with a time-varying covariate. It does thereby not matter whether the covariate originates from a continuous or dichotomous process or is just a deterministic time trend, and whether it is included as contemporaneous or lagged. The implications of this (additional) covariance and correlation are a matter of gradation depending on the sizes of the involved parameters. Our analysis for AR(1) covariates indicates that severe correlations (e.g. $r_{x_{t-1}y_{t-1}} > .7$) will only occur in cases where the true effects underlying the data (β_1, β_2, and particularly γ) are relatively large (see Figure 1). In practice, the issue will therefore not always be problematic, although it will still impact estimation precision. To shed further light on when which correlations can be encountered, we will simulate and study different types of covariates in the next section.

Simulation study

Goals

We conducted this simulation study with two goals in mind. Firstly we intended to verify our analytic results and showcase that predictor collinearity in AR(1) models with a lagged covariate generalizes beyond continuous covariates (with an AR(1) structure). We also wanted to illustrate how predictor collinearity depends on the amount of serial dependence in the covariate as well as on the size of β_1 and β_2. To this end, we simulated three different types of covariates

[6]We are thus assuming a specific restricted VAR(1) model in which the lagged effect of the outcome on the covariate is restricted to 0.

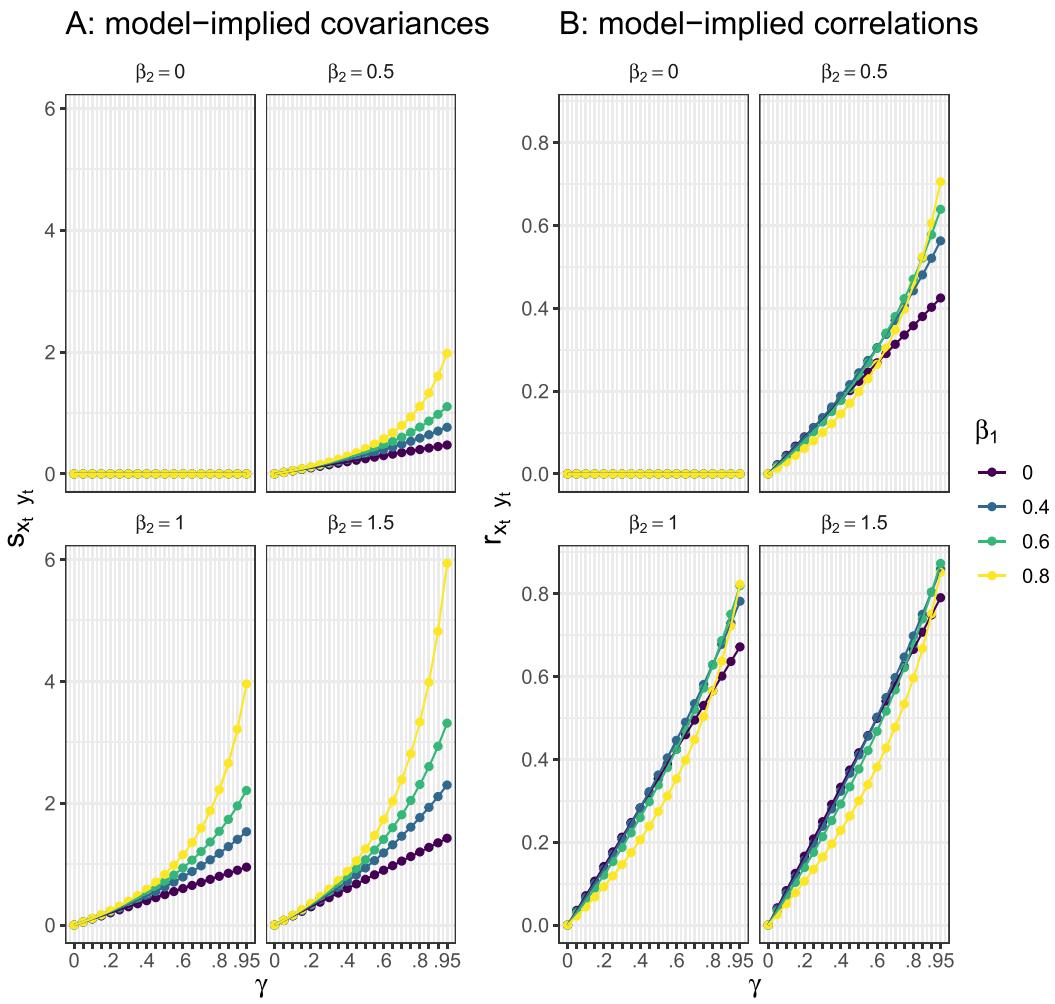


Figure 1. Model-implied predictor covariances (panel A) and predictor correlations (panel B) in case the lagged covariate is generated by an AR(1) process. The covariances and correlations are depicted as a function of the autocorrelation in the covariate (γ), various levels of β_1 (colors) and levels of β_2 (subpanels).

and manipulated their serial dependence, as well as β_1 and β_2.

Secondly we aimed to illustrate the impact of predictor collinearity on estimation precision and the extent to which this impact can be alleviated by increasing the number of time points. We predicted that estimation precision would decrease with higher predictor collinearity, but would benefit from larger numbers of time points. All data generation and analysis was performed using R version 4.0.4 (R Core Team, 2021).

Finally, although we were mainly interested in estimation precision, we also assessed possible biases in the estimators $\hat{\beta}_1$ and $\hat{\beta}_2$. As mentioned above, OLS results in *asymptotically* unbiased estimates of the parameters, but in small samples it is well known that the estimates of the autoregressive effect $\hat{\beta}_1$ are biased downwards (Maeshiro, 2000). The extent of this bias has been shown to become negligible for longer time series (Krone et al., 2017).

Simulation design and analysis

We simulated data from the AR(1) model with a time-varying lagged covariate:

$$y_t = \beta_0 + \beta_1 y_{t-1} + \beta_2 x_{t-1} + \varepsilon_t,$$

fixing β_0 to 1 and independently drawing all ε_t values from a standard normal distribution. We fully crossed four factors:

1. The number of time points, T, with six levels: 50, 100, 150, 250, 500, 1000;
2. the size of the AR effect of the lagged outcome, β_1, with three levels: 0, .4, .8;
3. the size of the effect of the lagged covariate, β_2, with three levels: .5, 1, 1.5;
4. the type of covariate x_t and the degree of serial dependence included, with 14 levels; see below.

Within each of the 756 resulting cells of the design, we generated 1000 datasets, by inserting the obtained

covariate values as well as β_1 and β_2 in the model equation above. In all cases, the initial value y_0 was drawn from the distribution of ε_t, and a burn-in period consisting of 500 time points was used.

Generating covariates

To assess the consequences of serially dependent covariates and to demonstrate that these consequences generalize to a wide range of possible covariates, we simulated three different types of covariates. Firstly, we generated covariates according to an AR(1) process (accommodating the case discussed in section "Expressions when the covariate follows an AR(1) process"), such that $x_t = \gamma x_{t-1} + \zeta_t$ where $\zeta_t \overset{iid}{\sim} N(0, \sigma_\zeta^2)$ and with uncorrelated innovations ζ_t and ε_t. We fixed $\sigma_\zeta^2 = 1$ and varied $\gamma = \{0, .2, .4, .6, .8, .9, .95\}$. Afterwards, the obtained x_t scores were standardized. Since increasing γ also increases the variability in the covariate, if σ_ζ^2 is kept constant, this standardization allows us to isolate the consequences of increasing serial dependence in the covariate.

Secondly, we generated dichotomous covariates setting p_1^x, the relative frequency of x_t taking on a value of 1, to .5. To vary the amount of serial dependence in the covariates, we manipulated the conditional probability $p_{1|1}^x$ of observing $x_t = 1$ given that $x_{t-1} = 1$. This conditional probability is a discrete counterpart of the autocorrelation, as discussed in Bodner et al. (2020) and termed autodependence in the following. Noting that $p_{1|0}^x$, which is the probability of switching from $x_{t-1} = 0$ to $x_t = 1$, can be approximated by:

$$p_{1|0}^x \approx \frac{p_1^x - p_1^x p_{1|1}^x}{1 - p_1^x}, \tag{18}$$

we conclude that the more $p_{1|0}^x$ and $p_{1|1}^x$ deviate from p_1^x, the stronger the autodependence in the covariate. Therefore, we varied $p_{1|1}^x = \{.5, .6, .7, .8, .9, .95\}$. In this case we did not standardize the covariate, to mimic typical dummy-coding of dichotomous predictors in applied research.

Finally, we simulated covariates that reflect a simple linear trend over time, implying an autocorrelation of one. Specifically, we generated $x_t = \frac{t}{T}$, and again standardized the resulting scores.

Analysis

To verify our analytic results, we first calculated for each generated dataset the predictor variances (s_y^2 and s_x^2) and covariance ($s_{x_{t-1}\, y_{t-1}}$). Next, we computed the average values of these statistics within each design cell, and used these to obtain an average predictor correlation. Note that this is the proper approach to obtain average correlations, since (independent) covariances are additive, and can thus be averaged, while correlations are not. For the AR(1) covariate case, where we could derive expressions as a function of the data-generating model parameters, we can thus directly assess whether the model-implied predictor covariances and correlations adequately describe their counterparts as observed in the simulated (small) samples.

To inspect the impact of time series length and predictor covariance and correlation on estimation precision, we fitted an AR(1) model with a lagged covariate effect to each generated dataset using OLS by means of the lm() function in R base. We computed the variances and correlation of the resulting estimates $\hat{\beta}_1$ and $\hat{\beta}_2$ within each design cell. Afterwards, the square roots of the average estimator variances per design cell were taken, to obtain average standard errors. Furthermore, we calculated absolute bias for the estimators $\hat{\beta}_1$, $\hat{\beta}_2$ as the average difference between the true and estimated parameter values over replications per design cell. We also obtained relative biases by dividing absolute biases by the respective true parameter value to provide an indication of the proportional extent of the biases.

Results

Serial dependence and predictor collinearity

Our first goal was to verify our analytic findings linking serial dependence in the covariate and predictor collinearity. Figure 2 displays the observed predictor covariances (left) and correlations (right) for the AR(1) covariates (Panels A), dichotomous covariates (Panels B), and the trending covariates (Panels C). The covariances and correlations are marked by colored circles with colors reflecting different sizes of β_1 and circle sizes reflecting the different time series lengths. For the AR(1) covariates, the simulation results are complemented by the model-implied predictor covariances and correlations, marked by red crosses. The simulation results confirm that if the serial dependence in the covariate (γ or $p_{1|1}^x$) increases, so do the observed predictor covariances and correlations. This confirms that the impact of serial dependence in the covariate on predictor collinearity generalizes over different types of covariates. However, Panels A1 and A2 also reveal discrepancies between the analytical and the observed predictor covariances and correlations, especially for shorter time series. These discrepancies are simply due to the observed (co)variances underestimating the population-level (co)variances, since they

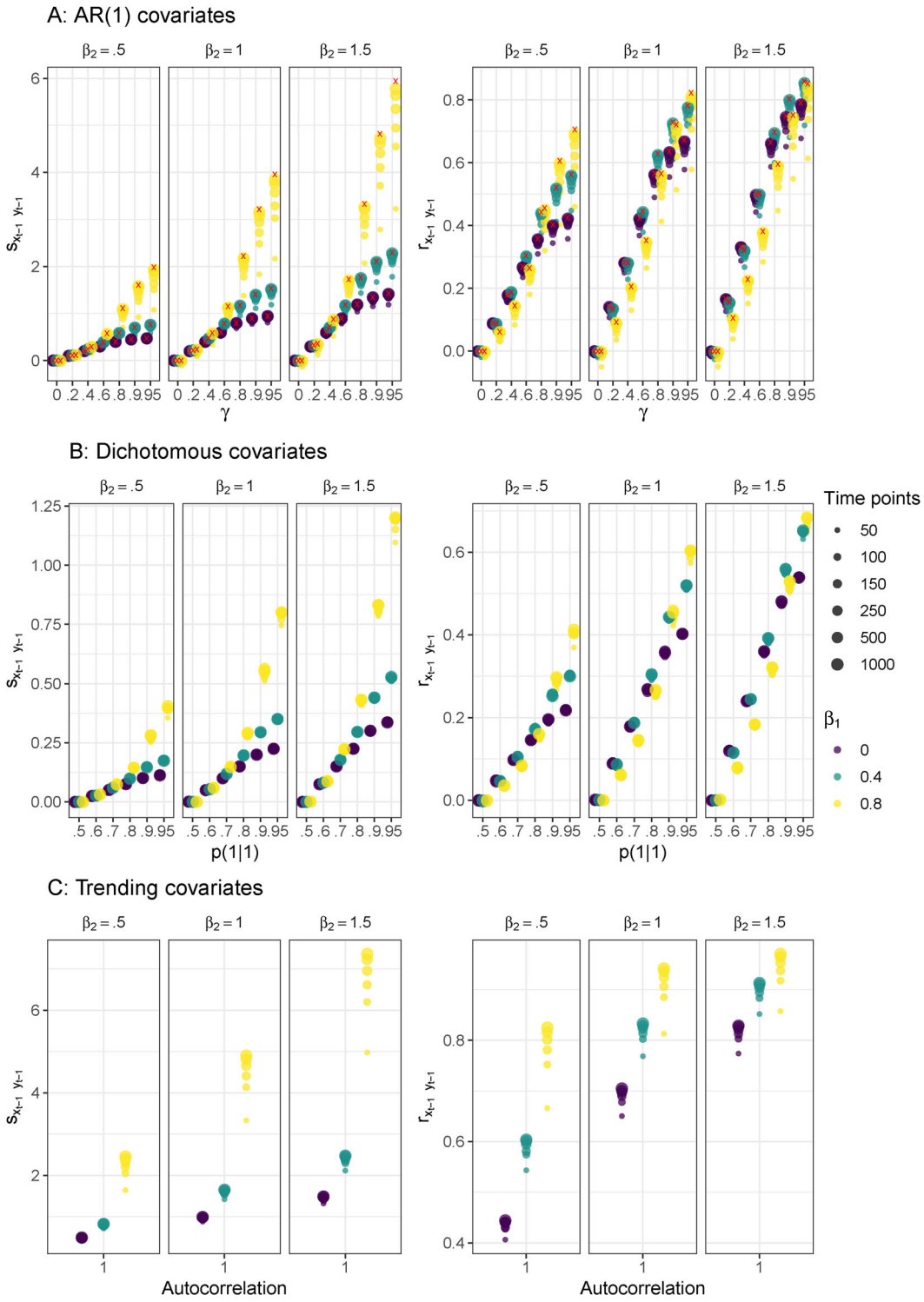


Figure 2. Observed predictor covariances (left) and correlations (right) for the simulated datasets as a function of parameter settings for β_2 (subplots), β_1 (colors), and time points (circle sizes). Panel A displays these statistics for AR(1) covariates, with γ on the x-axis. Red crosses correspond to the model-implied relationships for the AR(1) covariate with uncorrelated innovations (expression 17). Panel B displays these statistics for dichotomous covariates, with $p(1|1)$ on the x-axis. Panel C provides the statistics for the linear trend condition.

are calculated relative to the observed means rather than the population-level means (an illustration of this well-known problem can be found in (Bishop, 2006, p. 28)).

Estimation precision and small sample biases

Our second aim was to investigate the impact of predictor collinearity on estimation precision, and the role of the time series length therein. Figure 3 shows

that, as expected, the observed predictor correlations are strongly negatively related to the observed estimator correlations, for all covariate types. In addition, Figure 4 displays the expected increase in the estimator standard errors as a function of the serial dependence in the covariate (γ or $p^{x}_{1|1}$). As the serial dependence becomes larger, standard errors tend to increase, and this appears most pronounced for the estimate of the covariate effect, $\hat{\beta}_2$. Interestingly, large values of β_1 appear to be associated with *smaller* standard errors for $\hat{\beta}_1$ as γ increases, but for smaller values of β_1 we still observe a slight increase. In panel B, we observe similar relationships for dichotomous covariates. Panel C shows that the pattern of increasing standard errors as a function of serial dependence in the covariate that we saw for the AR covariates (panel A) extends to the case where the covariate is perfectly autocorrelated, as in our trending covariate conditions.

Regarding the role of the time series length, Figure 4 displays the observed estimator standard errors for time series of 50 and 1000 time points by means of different linetypes (Panels A and B) and along the x-axis (Panel C) respectively. It is evident that the combination of small samples and strongly serially dependent covariates constitute the most problematic cases. For example, for the trending covariate and $T = 50$, $\beta_1 = 0.4$, and $\beta_2 = 1$, the empirical standard error of $\hat{\beta}_2$ is larger than that for non-autocorrelated covariates (i.e. AR covariate where $\gamma = 0$) by a factor of 1.73.

Finally, Figure 5 reveals for small samples, we recover the expected downwards-bias for $\hat{\beta}_1$. We also observe a positive bias in $\hat{\beta}_2$, which does not appear to be present if $\gamma = 0$ (panel A) or $p(1|1) = 0$ (panel B), but increases as γ and $p(1|1)$ increase, suggesting that the bias in $\hat{\beta}_2$ originates due to correlation with the biased $\hat{\beta}_1$. As expected, the biases disappear in the large T condition. The magnitude of the biases for $T = 50$ are relatively small, but increase as a function of the autodependence in the covariate. For trending covariates (panel C), the small-sample bias in $\hat{\beta}_2$ can be quite pronounced. For example, the relative biases of $\hat{\beta}_2$ for $T = 50$, $\beta_1 = .4$, over sizes of $\beta_2 = \{0.5, 1, 1.5\}$ amount to .12, .11, and .08 respectively. For $\beta_1 = .8$, these relative biases amount to .34, .20, and .12, respectively.

Discussion

While the sources and consequences of predictor collinearity have received ample attention for cross-sectional methods (e.g. Mason & Perreault, 1991, Dormann et al., 2013), we studied the issue in the context of time series analysis, where it has been investigated far less (but see Bulteel et al., 2016, Bulteel et al., 2018). Specifically, this paper examined whether predictor collinearity in AR(1) models with a time-varying covariate can emerge as a function of the serial dependence within the variables involved. Our most important finding is that serial dependence in the covariate, an attribute which is usually not taken into account in applied research, induces (additional) covariance and correlation between the predictors (and hence the lagged outcome and covariate). The severity and thus consequences of such collinearity for applied research - lower estimation precision in the form of larger standard errors and stronger estimator tradeoffs - will vary strongly as a function of the various effect sizes underlying the data, and the number of time points at the researchers' disposal. Severe predictor collinearity will typically emerge if AR(1) models contain a covariate that is strongly autocorrelated or trending, and which has a large effect on the process. The decrease in estimation precision will be more pronounced in situations where the model is estimated on a smaller number of time points.The estimator that appears most prone to increased standard errors is the effect of the covariate itself ($\hat{\beta}_2$). Our simulation results suggest that the small sample bias (in our case $T = 50$) for $\hat{\beta}_1$ is associated with a compensatory bias in $\hat{\beta}_2$, the extent of which depends on $r_{\hat{\beta}_1\hat{\beta}_2}$ (and hence indirectly on the serial dependence in the covariate). The remainder of this paper will be dedicated to a number of implications of these findings for applied researchers, focusing on optimal study design and on extensions of the AR(1) model that we studied.

Implications for study design

We want to start by emphasizing that our results should not be interpreted as a recommendation to omit time-varying covariates purely due to concerns about collinearity. If a covariate indeed affects the process under study, failing to include it will lead to biased estimates (e.g. De Boef & Keele, 2008). Then the question obviously becomes how to best handle the issue uncovered. We think the answer lies in optimizing study design, rather than in changing the analysis strategy.

Firstly, one can think about the temporal properties of the time-varying covariate of interest. In experimental studies, one typically has control over this covariate, and manipulates it to produce variation in

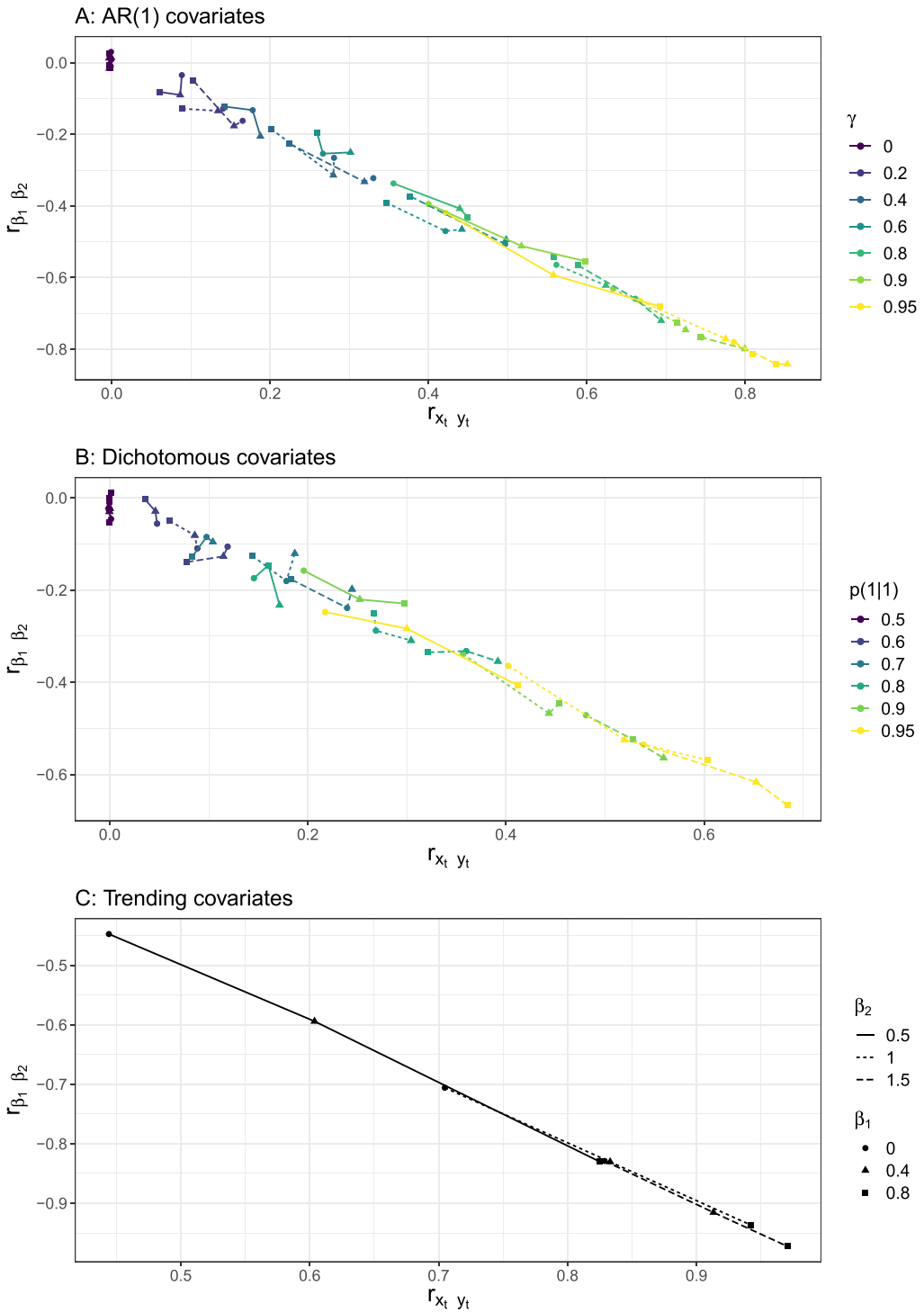


Figure 3. Relationships between the observed predictor correlations (x-axis) and observed estimator correlations (y-axis) as a function of serial dependence in the covariate (γ or $p(1|1)$) for the AR(1) covariate case (panel A) and dichotomous covariate case (panel B) respectively. Panel C provides the relationship for the trending covariate condition with time series length on the x-axis. Settings for β_2 and β_1 are given by line types and dot-shapes respectively.

the process of interest. Our results suggest to present experimental stimuli as *randomly* and void of serial dependence as possible, if permitted by the research question. In such situations, regardless of the magnitude of the effects of interest such as β_2 or β_1, the collinearity problem discussed here will not emerge. For observational studies (e.g., experience sampling studies), this straightforward recommendation is less applicable, since one typically has far less control over the variables under investigation.

In such cases, one can think of other ways to ensure that the effects can be estimated accurately.

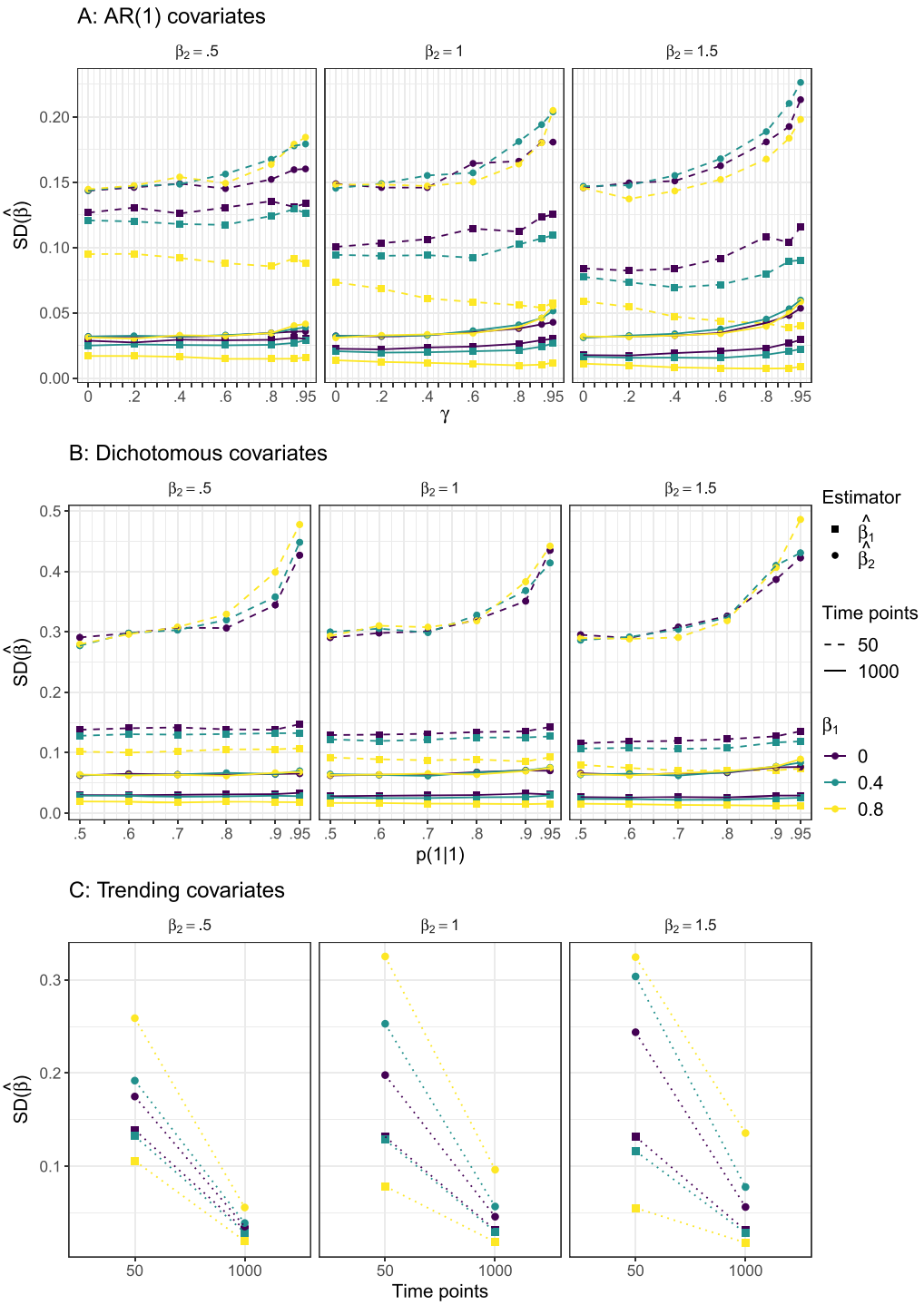


Figure 4. Observed standard deviations (i.e., standard errors) of the estimators (y-axis) $\hat{\beta}_1$ (squares) and $\hat{\beta}_2$ (dots) as a function of serial dependence in the covariate (γ or $p(1|1)$) for the AR(1) covariate case (panel A) and dichotomous covariate case (panel B) respectively. Settings for β_1 are indicated by colors, and the number of simulated time points is indicated by the line type, with solid lines corresponding to the small (T = 50) and large (T = 1000) data sets. Subplots correspond to settings for β_2. Panel C shows the observed estimator standard errors for small and large simulated data sets (x-axis) for the trend condition.

Since a larger number of time points leads to increased estimation precision, collecting more observations would help in mitigating the consequences of predictor collinearity. Indeed, we saw in the simulation study that including more time points reduces the standard errors of the estimators, also in case of strong serial dependence, and thus also reduces the practical consequences of estimator tradeoffs (i.e., high estimator correlations play out less in terms of unstandardized parameter estimates). Since estimation

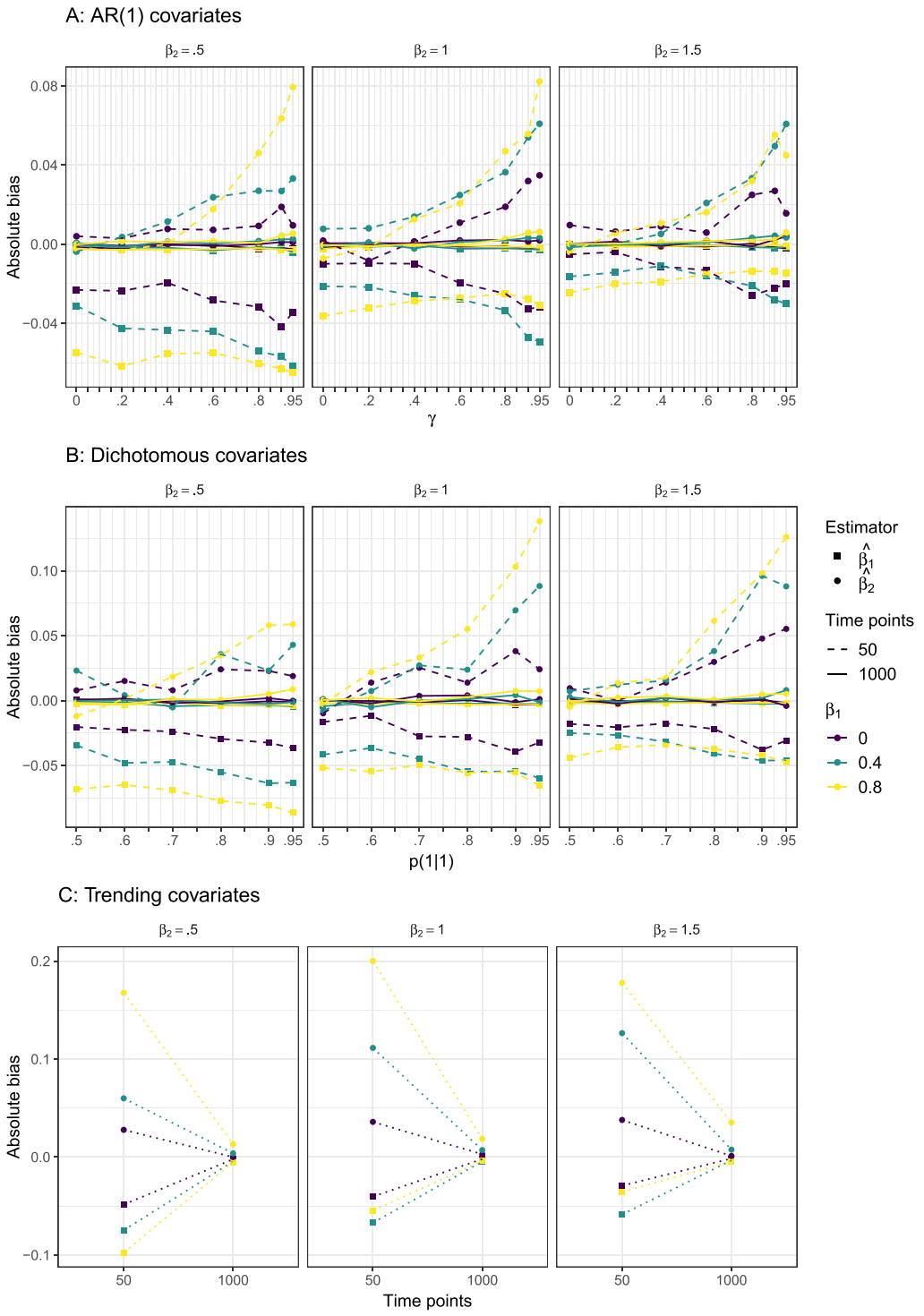


Figure 5. Observed absolute bias of the estimators $\hat{\beta}_1$ (squares) and $\hat{\beta}_1$ (dots) displayed on the y-axis as a function of the serial dependence in the covariate (i.e., γ or $p(1|1)$). Panel A displays the results for the AR covariates, and Panel B for the dichotomous covariates. Panel C shows the observed absolute bias for small and large conditions (x-axis) for the trend condition. Settings for β_1 are indicated by colors, and the number of simulated time points is indicated by the line type (in Panels A and B), with solid lines corresponding to large (T = 1000) and dashed lines corresponding to the small (T = 50) simulation conditions. Subplots correspond to different settings for β_2.

precision is related to power (Brandmaier et al., 2018), this also means that our results have immediate implications for how to run power analyses for the models under study. For instance, Lafit et al. (2021) recently discussed a simulation-based power analysis approach to optimize sample size when one aims to fit multilevel models including AR(1) terms and time-varying covariates. When specifying the data-generating

models that are used in the simulation step one should make sure that potential serial dependence in the covariate is taken into account and included when generating data. Otherwise, power calculations might be inaccurate.

Implications for model extensions

Moderating the autoregressive effect

We mentioned in the introduction that, next to modeling intercept changes, many researchers justly desire a way of quantifying changes in the AR effect over time, for instance to capture and understand fluctuations in emotional inertia (e.g. Koval & Kuppens, 2012). To this end, researchers can fit a model that contains an interaction effect ($\beta_3 x_{t-1} y_{t-1}$) in addition to the main effects of predictors x_{t-1} and y_{t-1}. While we conjecture that this interaction model will also be prone to the collinearity issues discussed in this paper, since it contains our focus model as a special case, the interaction model deserves further attention. Research on cross-sectional methods (e.g., McClelland et al., 2017) indicates that predictor collinearity issues become more complex in such models (e.g., large predictor correlations can sometimes be beneficial for the estimation of the interaction term; McClelland & Judd, 1993, Shieh, 2010) and depend, amongst others, on whether or not these predictors are centered (Cohen et al., 2013, p.246, Smith & Sasaki, 1979). Since centering in AR(1) models is a topic in its own right (E. L. Hamaker & Grasman, 2014) and since researchers usually aim to report and interpret not only the effect of the covariate on the autoregressive effect, β_3, but also β_1 and β_2, the additional impact of serial dependence on the estimation of all three regression weights clearly deserves further study.

Time polynomials

Our simulations clearly demonstrated that including time as a linear trend in AR(1) models, which is a popular practice to control for time trends, may lead to collinearity issues, that considerably lower estimation precision when the number of time points is on the small side. In practice, the issue might even get worse, as many researchers try to further control for the often 'wobbly' nature of mean changes in time series by including additional polynomials of time (e.g. $t, t^2, t^3, \ldots$) as predictors in the model. It is known that polynomial regression models in general suffer from collinearity issues as the polynomial order increases (Shacham & Brauner, 1997). Combining these findings with our results, we hypothesize that models with additional time polynomials will only be more vulnerable to collinearity issues of the here-discussed nature. It might therefore be wise to follow the advice of Michaelides and Spanos (2020), who argue that a two-stage procedure, in which the regression on time is done prior to estimating the effects of the other predictors, might be a more appropriate way for controlling for such changes over time. Comparing both approaches, two-step and simultaneous, is a useful direction for further research.

Distributed lags and VAR models

Next to investigating AR(1) models that include interaction terms or time polynomials, it would also be fruitful to consider extensions of AR(1) models that include multiple effects of the process variable(s) on themselves. Here, we think of AR(p) models, which allow to investigate the effect of the previous p process states, as well as VAR(1) models for multivariate processes and the AR distributed lag model (ADL, see Wooldridge, 2012, Harvey, 1989), which includes both lagged and contemporaneous covariate effects. Indeed, both the lagged and contemporaneous models without innovation covariances discussed in this paper could be seen as special cases of ADL models. We also note that the lagged model with an AR(1) covariate corresponds to a restricted VAR(1) model.

Multi-subject models

Whereas this paper has been concerned with the $N = 1$ case, many researchers wish to investigate inter-individual differences in intra-individual dynamics and relate these to person-level variables in one way or another. To this end they often apply multilevel or mixed models, where the level 1 equation can also boil down to an AR(1) process with time-varying covariates (e.g. Gistelinck & Loeys, 2020, Lafit et al., 2021, Koval & Kuppens, 2012). Studying the potential repercussions of serial dependence in the covariate for such models, seems highly useful. Indeed, if it would turn out that estimating the included random AR and covariate effects would be troublesome in case of high serial dependence in the covariate, this would clearly be important.

Conclusion

While AR(1) models that include time-varying covariates are often used in empirical research, researchers may be unaware of the predictor collinearity that can emerge due to serial dependence in the covariate in combination with moderate to strong AR and

covariate effects. The practical consequences of the collinearity are strongly dependent on the number of time points used to estimate the models, however. Therefore, it is always prudent to invest in sufficient time points, and sample size planning should take the (expected) serial dependence in the covariate of interest into account.

Article information

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Ethical Principles: The authors affirm having followed professional ethical guidelines in preparing this work. These guidelines include obtaining informed consent from human participants, maintaining ethical treatment and respect for the rights of human or animal participants, and ensuring the privacy of participants and their data, such as ensuring that individual participants cannot be identified in reported results or from publicly available original or archival data.

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Appendix

A. Covariance matrix of the OLS estimators for the AR(1) model with a time-varying lagged covariate

In this appendix, we derive the asymptotic covariance matrix of the OLS estimators under the AR(1) model with a time-varying lagged covariate:

$$y_t = \beta_0 + \beta_1 y_{t-1} + \beta_2 x_{t-1} + \epsilon_t. \tag{A.19}$$

For this model, the design matrix $\boldsymbol{X}$ contains a column of 1's, a column of lagged-outcome values, and a column of lagged covariate values:

$$\boldsymbol{X} = \begin{bmatrix} 1 & y_0 & x_0 \\ 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ \vdots & \vdots & \vdots \\ 1 & y_{T-1} & x_{T-1} \end{bmatrix}. \tag{A.20}$$

Given that OLS estimators cannot handle missing predictor values, there are different strategies of obtaining the initial values y_0 and x_0. One might for instance omit the first observation of the outcome variable when fitting the model or impute values for the predictors. For somewhat longer time series, however, it will not have much of an effect which approach one chooses.

Generally, we know that for any $T \times (P+1)$ design matrix (Ledoit & Wolf, 2003):

$$\boldsymbol{X}'\boldsymbol{X} = T\left(\boldsymbol{S}_{\boldsymbol{x}} + \boldsymbol{m}_{\boldsymbol{x}}\boldsymbol{m}_{\boldsymbol{x}}'\right). \tag{A.21}$$

This implies that each individual element $\{m, n\}$ of $\boldsymbol{X}'\boldsymbol{X}$ equals

$$\begin{aligned} \boldsymbol{X}'\boldsymbol{X}_{m,n} &= \boldsymbol{x}^{m\prime}\boldsymbol{x}^{n} \\ &= \sum_{t=1}^{T} x_t^m\, x_t^n \\ &= x_1^m\, x_1^n + x_2^m\, x_2^n + \cdots + x_T^m\, x_T^n \\ &= T\left(s_{x^m x^n} + m_{x^m} m_{x^n}\right), \end{aligned} \tag{A.22}$$

where $\boldsymbol{x}^m$ is the m-th column vector of design matrix $\boldsymbol{X}$, $\boldsymbol{x}^n$ is the n-th column vector, $s_{x^m x^n}$ is the covariance of (the values in the) vectors $\boldsymbol{x}^m$ and $\boldsymbol{x}^n$ and m_{x^m} and m_{x^n} are their arithmetic means.

For the above-considered model, matrix $\boldsymbol{X}'\boldsymbol{X}$ thus looks as follows:

$$\boldsymbol{X}'\boldsymbol{X} = \begin{bmatrix} T & Tm_{y_{t-1}} & Tm_{x_{t-1}} \\ Tm_{y_{t-1}} & T(s_{y_{t-1}}^2 + m_{y_{t-1}}^2) & T(s_{y_{t-1}x_{t-1}} + m_{y_{t-1}} m_{x_{t-1}}) \\ Tm_{x_{t-1}} & T(s_{y_{t-1}x_{t-1}} + m_{y_{t-1}} m_{x_{t-1}}) & T(s_{x_{t-1}}^2 + m_{x_{t-1}}^2) \end{bmatrix}. \tag{A.23}$$

Writing the covariance of vectors $\boldsymbol{y}_{t-1}$ and $\boldsymbol{x}_{t-1}$ as a function of standard deviations and the correlation, we get:

$$\boldsymbol{X}'\boldsymbol{X} = \begin{bmatrix} T & Tm_{y_{t-1}} & Tm_{x_{t-1}} \\ Tm_{y_{t-1}} & T(s_{y_{t-1}}^2 + m_{y_{t-1}}^2) & T(r_{y_{t-1}x_{t-1}} s_{y_{t-1}} s_{x_{t-1}} + m_{y_{t-1}} m_{x_{t-1}}) \\ Tm_{x_{t-1}} & T(r_{y_{t-1}x_{t-1}} s_{y_{t-1}} s_{x_{t-1}} + m_{y_{t-1}} m_{x_{t-1}}) & T(s_{x_{t-1}}^2 + m_{x_{t-1}}^2) \end{bmatrix}. \tag{A.24}$$

Via the relationship $\boldsymbol{\Sigma}_{\hat{\boldsymbol{\beta}}} = \sigma_\epsilon^2 (\boldsymbol{X}'\boldsymbol{X})^{-1}$ and using Mathematica for symbolic matrix inversion (Wolfram Research & Inc, 2020) we get the associated covariance matrix of the OLS estimators $\hat{\boldsymbol{\beta}} = (\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2)'$ as:

$$\boldsymbol{\Sigma}_{\hat{\boldsymbol{\beta}}} = \begin{bmatrix} \sigma_{\hat{\beta}_0}^2 & \sigma_{\hat{\beta}_0\hat{\beta}_1} & \sigma_{\hat{\beta}_0\hat{\beta}_2} \\ \sigma_{\hat{\beta}_0\hat{\beta}_1} & \sigma_{\hat{\beta}_1}^2 = \dfrac{\sigma_\epsilon}{T\left(1 - r_{y_{t-1}x_{t-1}}^2\right) s_{y_{t-1}}^2} & \sigma_{\hat{\beta}_1\hat{\beta}_2} = -\dfrac{r_{y_{t-1}x_{t-1}}\sigma_\epsilon}{T\left(1 - r_{y_{t-1}x_{t-1}}^2\right) s_{y_{t-1}} s_{x_{t-1}}} \\ \sigma_{\hat{\beta}_0\hat{\beta}_2} & \sigma_{\hat{\beta}_1\hat{\beta}_2} = -\dfrac{r_{y_{t-1}x_{t-1}}\sigma_\epsilon}{T\left(1 - r_{y_{t-1}x_{t-1}}^2\right) s_{y_{t-1}} s_{x_{t-1}}} & \sigma_{\hat{\beta}_2}^2 = \dfrac{\sigma_\epsilon}{T\left(1 - r_{y_{t-1}x_{t-1}}^2\right) s_{x_{t-1}}^2} \end{bmatrix} \tag{A.25}$$

with the following expressions for the entries for the variance and covariances of the intercept estimator:

$$\sigma_{\hat{\beta}_0}^2 = -\frac{\sigma_\epsilon\left(-2r_{y_{t-1}x_{t-1}} m_{y_{t-1}} m_{x_{t-1}} s_{y_{t-1}} s_{x_{t-1}} + s_{x_{t-1}}^2\left(m_{y_{t-1}}^2 - \left(r_{y_{t-1}x_{t-1}}^2 - 1\right) s_{y_{t-1}}^2\right) + m_{x_{t-1}}^2 s_{y_{t-1}}^2\right)}{T\left(r_{y_{t-1}x_{t-1}}^2 - 1\right) s_{y_{t-1}}^2 s_{x_{t-1}}^2} \tag{A.26}$$

$$\sigma_{\hat{\beta}_0\hat{\beta}_1} = \frac{\sigma_\epsilon\left(m_{y_{t-1}} s_{x_{t-1}} - r_{y_{t-1}x_{t-1}} m_{x_{t-1}} s_{y_{t-1}}\right)}{T\left(r^2_{y_{t-1}x_{t-1}} - 1\right) s^2_{y_{t-1}} s_{x_{t-1}}} \tag{A.27}$$

$$\sigma_{\hat{\beta}_0\hat{\beta}_2} = \frac{\sigma_\epsilon\left(m_{x_{t-1}} s_{y_{t-1}} - r_{y_{t-1}x_{t-1}} m_{y_{t-1}} s_{x_{t-1}}\right)}{T\left(r^2_{y_{t-1}x_{t-1}} - 1\right) s_{y_{t-1}} s^2_{x_{t-1}}}. \tag{A.28}$$

The correlation between the regression weight estimators can then be obtained as:

$$\begin{aligned}
\rho_{\hat{\beta}_1\hat{\beta}_2} &= \frac{\sigma_{\hat{\beta}_1\hat{\beta}_2}}{\sqrt{\sigma^2_{\hat{\beta}_1}}\sqrt{\sigma^2_{\hat{\beta}_2}}} \\
&= -r_{y_{t-1}x_{t-1}} \frac{\sigma_\epsilon}{T\left(1 - r^2_{y_{t-1}x_{t-1}}\right) s_{y_{t-1}} s_{x_{t-1}} \sqrt{\frac{\sigma_\epsilon}{T\left(1 - r^2_{y_{t-1}x_{t-1}}\right) s^2_{y_{t-1}}}} \sqrt{\frac{\sigma_\epsilon}{T\left(1 - r^2_{y_{t-1}x_{t-1}}\right) s^2_{x_{t-1}}}}} \\
&= -r_{y_{t-1}x_{t-1}} \frac{\sigma_\epsilon}{T\left(1 - r^2_{y_{t-1}x_{t-1}}\right) s_{y_{t-1}} s_{x_{t-1}} \sqrt{\sigma_\epsilon} \sqrt{\frac{1}{T\left(1 - r^2_{y_{t-1}x_{t-1}}\right)}} \sqrt{\frac{1}{s^2_{y_{t-1}}}} \sqrt{\sigma_\epsilon} \sqrt{\frac{1}{T\left(1 - r^2_{y_{t-1}x_{t-1}}\right)}} \sqrt{\frac{1}{s^2_{x_{t-1}}}}} \\
&= -r_{y_{t-1}x_{t-1}} \frac{\sigma_\epsilon}{T\left(1 - r^2_{y_{t-1}x_{t-1}}\right) s_{y_{t-1}} s_{x_{t-1}} \frac{\sigma_\epsilon}{s_{y_{t-1}} s_{x_{t-1}} T\left(1 - r^2_{y_{t-1}x_{t-1}}\right)}} \\
&= -r_{y_{t-1}x_{t-1}} \frac{\sigma_\epsilon s_{y_{t-1}} s_{x_{t-1}} T\left(1 - r^2_{y_{t-1}x_{t-1}}\right)}{T\left(1 - r^2_{y_{t-1}x_{t-1}}\right) s_{y_{t-1}} s_{x_{t-1}} \sigma_\epsilon} \\
&= -r_{y_{t-1}x_{t-1}}
\end{aligned} \tag{A.29}$$

B. Decomposing the covariance of the predictors

B.1. The general case of any time-varying lagged covariate

In this appendix, we further decompose the covariance between the predictor vectors $\boldsymbol{y}_{t-1}$ and $\boldsymbol{x}_{t-1}$, $s_{y_{t-1}x_{t-1}}$. Using Equation A.22, we can write:

$$\begin{aligned}
\boldsymbol{y}'_{t-1}\boldsymbol{x}_{t-1} &= T\left(s_{y_{t-1}x_{t-1}} + m_{y_{t-1}} m_{x_{t-1}}\right) \\
\iff \quad s_{y_{t-1}x_{t-1}} &= \frac{1}{T}\boldsymbol{y}'_{t-1}\boldsymbol{x}_{t-1} - m_{y_{t-1}} m_{x_{t-1}} \\
s_{y_{t-1}x_{t-1}} &= \frac{1}{T}\boldsymbol{y}'_{t-1}\boldsymbol{x}_{t-1} - \frac{1}{T}(\boldsymbol{y}'_{t-1}\,\mathbf{1})\frac{1}{T}(\boldsymbol{x}'_{t-1}\,\mathbf{1}),
\end{aligned} \tag{B.30}$$

where $\mathbf{1}$ is a column vector of 1's. Without loss of generality we assume that $\boldsymbol{x}_{t-1}$ is centered so that the above expression simplifies to:

$$s_{y_{t-1}x_{t-1}} = \frac{1}{T}\boldsymbol{y}'_{t-1}\boldsymbol{x}_{t-1}. \tag{B.31}$$

We now replace $\boldsymbol{y}'_{t-1}$ with $(\beta_0\mathbf{1} + \beta_1\boldsymbol{y}_{t-2} + \beta_2\boldsymbol{x}_{t-2} + \boldsymbol{\epsilon}_{t-1})'$ and solve:

$$\begin{aligned}
s_{y_{t-1}x_{t-1}} &= \frac{1}{T}\boldsymbol{y}'_{t-1}\boldsymbol{x}_{t-1} \\
&= \frac{1}{T}\left(\beta_0\mathbf{1} + \beta_1\boldsymbol{y}_{t-2} + \beta_2\boldsymbol{x}_{t-2} + \boldsymbol{\epsilon}_{t-1}\right)'\boldsymbol{x}_{t-1} \\
&\text{|substitute} \\
&= \frac{1}{T}\left(\beta_0\mathbf{1}' + \beta_1\boldsymbol{y}'_{t-2} + \beta_2\boldsymbol{x}'_{t-2} + \boldsymbol{\epsilon}'_{t-1}\right)\boldsymbol{x}_{t-1} \\
&\text{|apply rules for transpose} \\
&= \frac{1}{T}\left(\beta_0\mathbf{1}'\boldsymbol{x}_{t-1} + \beta_1\boldsymbol{y}'_{t-2}\boldsymbol{x}_{t-1} + \beta_2\boldsymbol{x}'_{t-2}\boldsymbol{x}_{t-1} + \boldsymbol{\epsilon}'_{t-1}\boldsymbol{x}_{t-1}\right) \\
&\text{|distribute} \\
&= \frac{1}{T}\left(\beta_1\boldsymbol{y}'_{t-2}\boldsymbol{x}_{t-1} + \beta_2\boldsymbol{x}'_{t-2}\boldsymbol{x}_{t-1} + \boldsymbol{\epsilon}'_{t-1}\boldsymbol{x}_{t-1}\right) \\
&\text{|}\mathbf{1}'\boldsymbol{x}_{t-1} = 0 \text{ since covariate is centered} \\
&= \beta_1\frac{1}{T}\boldsymbol{y}'_{t-2}\boldsymbol{x}_{t-1} + \beta_2\frac{1}{T}\boldsymbol{x}'_{t-2}\boldsymbol{x}_{t-1} + \frac{1}{T}\boldsymbol{\epsilon}'_{t-1}\boldsymbol{x}_{t-1} \\
&= \beta_1 s_{y_{t-2}x_{t-1}} + \beta_2 s_{x_{t-2}x_{t-1}} + s_{\epsilon_{t-1}x_{t-1}}.
\end{aligned} \tag{B.32}$$

This expression still features a covariance term concerning both predictors, $s_{y_{t-2}x_{t-1}}$. To further decompose this term, we again use recursive substitution: In $s_{y_{t-2}x_{t-1}} = \frac{1}{T}\boldsymbol{y}'_{t-2}\boldsymbol{x}_{t-1}$, we replace $\boldsymbol{y}'_{t-2}$ with $(\beta_0\mathbf{1} + \beta_1\boldsymbol{y}_{t-3} + \beta_2\boldsymbol{x}_{t-3} + \boldsymbol{\epsilon}_{t-2})'$ and get:

$$s_{y_{t-2}x_{t-1}} = \beta_1 s_{y_{t-3}x_{t-1}} + \beta_2 s_{x_{t-3}x_{t-1}} + s_{\epsilon_{t-2}x_{t-1}} \tag{B.33}$$

Reinserting this into the last line of Equation B.32 yields:

$$\begin{aligned}
s_{y_{t-1}x_{t-1}} &= \beta_1\left(\beta_1 s_{y_{t-3}x_{t-1}} + \beta_2 s_{x_{t-3}x_{t-1}} + s_{\epsilon_{t-2}x_{t-1}}\right) + \beta_2 s_{x_{t-2}x_{t-1}} + s_{\epsilon_{t-1}x_{t-1}} \\
&= \beta_1^2 s_{y_{t-3}x_{t-1}} + \beta_1\beta_2 s_{x_{t-3}x_{t-1}} + \beta_2 s_{x_{t-2}x_{t-1}} + \beta_1 s_{\epsilon_{t-2}x_{t-1}} + s_{\epsilon_{t-1}x_{t-1}}.
\end{aligned} \tag{B.34}$$

Assuming that our AR(1) process with the covariate effect started a long time ago, and we were able to take sufficiently many observations, we can go an arbitrary number of K steps back in time:

$$\begin{aligned}
s_{y_{t-1}x_{t-1}} &= \beta_1^2 s_{y_{t-3}x_{t-1}} + \beta_1\beta_2 s_{x_{t-3}x_{t-1}} + \beta_2 s_{x_{t-2}x_{t-1}} + \beta_1 s_{\epsilon_{t-2}x_{t-1}} + s_{\epsilon_{t-1}x_{t-1}} \\
&= \beta_1^3 s_{y_{t-4}x_{t-1}} + \beta_1^2\beta_2 s_{x_{t-4}x_{t-1}} + \beta_1\beta_2 s_{x_{t-3}x_{t-1}} + \beta_2 s_{x_{t-2}x_{t-1}} \\
&\quad + \beta_1^2 s_{\epsilon_{t-3}x_{t-1}} + \beta_1 s_{\epsilon_{t-2}x_{t-1}} + s_{\epsilon_{t-1}x_{t-1}} \\
&\quad \vdots \\
&= \beta_1^K s_{y_{(t-1)-K}x_{t-1}} + \sum_{k=1}^{K}\left(\beta_1^{k-1}\beta_2 s_{x_{(t-1)-k}x_{t-1}} + \beta_1^{k-1} s_{\epsilon_{t-k}x_{t-1}}\right) \\
&= \beta_1^K s_{y_{(t-1)-K}x_{t-1}} + \sum_{k=1}^{K}\beta_1^{k-1}\left(\beta_2 s_{x_{(t-1)-k}x_{t-1}} + s_{\epsilon_{t-k}x_{t-1}}\right).
\end{aligned} \tag{B.35}$$

If K approaches ∞, β_1^K will approach zero since $|\beta_1| < 1$, and the expression simplifies to:

$$s_{y_{t-1}x_{t-1}} = \sum_{k=1}^{\infty}\beta_1^{k-1}\left(\beta_2 s_{x_{(t-1)-k}x_{t-1}} + s_{\epsilon_{t-k}x_{t-1}}\right). \tag{B.36}$$

Note that the presented steps can also be taken to derive the predictor covariance under the model with the contemporaneous covariate. Only the starting point is then a different one, namely $s_{y_{t-1}x_t} = \frac{1}{T}\boldsymbol{y}'_{t-1}\boldsymbol{x}_t$. For the contemporaneous covariate, we thus get:

$$s_{y_{t-1}x_t} = \beta_1 s_{y_{t-2}x_t} + \beta_2 s_{x_{t-1}x_t} + s_{\epsilon_{t-1}x_t}, \tag{B.37}$$

$$\begin{aligned}
&= \beta_1^K s_{y_{(t-1)-K}x_t} \\
&\quad + \sum_{k=1}^{K}\beta_1^{k-1}(\beta_2 s_{x_{t-k}x_t} + s_{\epsilon_{t-k}x_t}), \quad \text{| Go } K \text{ steps back in time} \\
&= \sum_{k=1}^{\infty}\beta_1^{k-1}(\beta_2 s_{x_{t-k}x_t} + s_{\epsilon_{t-k}x_t}). \qquad \text{|Let } K \text{ approach } \infty
\end{aligned}$$

B.2. The special case of an AR(1) covariate

We now assume that the values in $\boldsymbol{x}_t$ are generated by the zero-mean stationary AR(1) process

$$x_t = \gamma x_{t-1} + \zeta_t, \tag{B.38}$$

with $|\gamma| < 1$. For the innovations ζ_t we assume that they are jointly normally distributed with the innovations ϵ_t which feed into y_t (see Equation A.19):

$$\begin{bmatrix} \zeta_t \\ \epsilon_t \end{bmatrix} \overset{iid}{\sim} N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_\zeta^2 & \sigma_{\zeta\epsilon} \\ \sigma_{\zeta\epsilon} & \sigma_\epsilon^2 \end{bmatrix} \right). \tag{B.39}$$

The contemporaneous covariance between the innovations, $\sigma_{\zeta\epsilon}$, can thus be non-zero. Based on these assumptions, we can further simplify the infinite sum given in Equation B.36, as we will show in the following.

From the time series literature (Hamilton, 1994, p. 53), we know that the kth autocovariance of x_t equals:

$$\sigma_{x_t\, x_{t-k}} = \gamma^k \frac{\sigma_\zeta^2}{1-\gamma^2}. \tag{B.40}$$

This kth autocovariance is independent of t, so the expression also holds for $t-1$, for instance. Also, using the expected value operator and the Wold representation of the covariate process (i.e., an AR(1) process can be rewritten as a weighted sum of all its innovations up to time t and its mean, which is 0 in this case), we can derive the model-implied covariance of x_{t-1} with the preceding innovations ϵ_{t-k} as (in analogy to (Hamilton, 1994, p. 53)):

$$\begin{aligned} \sigma_{\epsilon_{t-k}\, x_{t-1}} &= E(\epsilon_{t-k}\, x_{t-1}) \\ &= E((\epsilon_{t-k})\,(\zeta_{t-1} + \gamma\,\zeta_{t-2} + \gamma^2\,\zeta_{t-3} + \ldots + \gamma^{k-1}\,\zeta_{t-k} + \ldots)) \\ &= \gamma^{k-1} E(\epsilon_{t-k}\,\zeta_{t-k}) \\ &= \gamma^{k-1}\sigma_{\zeta\,\epsilon} \end{aligned} \tag{B.41}$$

This means that, under the here-considered AR(1) model for the lagged covariate, the model-implied analogue to Equation B.36 can be written as

$$\begin{aligned} \sigma_{y_{t-1}x_{t-1}} &= \sum_{k=1}^{\infty} \beta_1^{k-1}\left(\beta_2\gamma^k\,\frac{\sigma_\zeta^2}{1-\gamma^2} + \gamma^{k-1}\sigma_{\zeta\,\epsilon}\right) \\ &= \sum_{k=1}^{\infty}\left(\beta_1^{k-1}\beta_2\gamma^k\,\frac{\sigma_\zeta^2}{1-\gamma^2}\right) + \sum_{k=1}^{\infty}\left(\beta_1^{k-1}\gamma^{k-1}\sigma_{\zeta\,\epsilon}\right) \\ &= \frac{\beta_2}{\beta_1}\,\frac{\sigma_\zeta^2}{1-\gamma^2}\sum_{k=1}^{\infty}(\beta_1\gamma)^k + \frac{\sigma_{\zeta\,\epsilon}}{\beta_1\,\gamma}\sum_{k=1}^{\infty}(\beta_1\,\gamma)^k \\ &= \left(\frac{\beta_2}{\beta_1}\,\frac{\sigma_\zeta^2}{1-\gamma^2} + \frac{\sigma_{\zeta\,\epsilon}}{\beta_1\,\gamma}\right)\sum_{k=1}^{\infty}(\beta_1\,\gamma)^k \end{aligned} \tag{B.42}$$

Since $|\beta_1| < 1$ and $|\gamma| < 1$, we know that the infinite sum $\sum_{k=1}^{\infty}(\beta_1\,\gamma)^k$ becomes $\frac{\beta_1\gamma}{1-\beta_1\gamma}$ (Weisstein (n.d.)), hence:

$$\begin{aligned} \sigma_{y_{t-1}x_{t-1}} &= \left(\frac{\beta_2}{\beta_1}\,\frac{\sigma_\zeta^2}{1-\gamma^2} + \frac{\sigma_{\zeta\,\epsilon}}{\beta_1\,\gamma}\right)\frac{\beta_1\,\gamma}{1-\beta_1\,\gamma} \\ &= \frac{\beta_2}{\beta_1}\,\frac{\sigma_\zeta^2}{1-\gamma^2}\,\frac{\beta_1\,\gamma}{1-\beta_1\,\gamma} + \frac{\sigma_{\zeta\,\epsilon}}{\beta_1\,\gamma}\,\frac{\beta_1\,\gamma}{1-\beta_1\,\gamma} \\ &= \beta_2\,\gamma\,\frac{\sigma_\zeta^2}{1-\gamma^2}\,\frac{1}{1-\beta_1\,\gamma} + \sigma_{\zeta\,\epsilon}\,\frac{1}{1-\beta_1\,\gamma} \\ &= \left(\beta_2\,\gamma\,\frac{\sigma_\zeta^2}{1-\gamma^2} + \sigma_{\zeta\,\epsilon}\right)\frac{1}{1-\beta_1\,\gamma}, \end{aligned} \tag{B.43}$$

where $\frac{\sigma_\zeta^2}{1-\gamma^2}$ is the model-implied variance of x_t, σ_x^2.

Finally, to additionally get an expression of the predictor correlation as a function of the model parameters, we still need to derive the model-implied variance of y_t as:

$$\sigma_y^2 = E(y_t\,y_t) - E(y_t)^2. \tag{B.44}$$

For $E(y_t)^2$ we get:

$$\begin{aligned} E(y_t)^2 &= E(\beta_0 + \beta_1\,y_{t-1} + \beta_2\,x_{t-1} + \epsilon_t)^2 \\ &= \left(\beta_0 + \beta_1\,E(y_{t-1}) + \beta_2\,E(x_{t-1}) + E(\epsilon_t)\right)^2 \\ &= (\beta_0 + \beta_1\,\mu_y)^2 \\ &= \beta_0^2 + 2\beta_0\,\beta_1\,\mu_y + \beta_1^2\,\mu_y^2, \end{aligned} \tag{B.45}$$

since x_t and ϵ_t are assumed to have zero means. For $E(y_t\,y_t)$ we get:

$$\begin{aligned} E(y_t\,y_t) &= E((\beta_0 + \beta_1\,y_{t-1} + \beta_2\,x_{t-1} + \epsilon_t)(\beta_0 + \beta_1\,y_{t-1} + \beta_2\,x_{t-1} + \epsilon_t)) \\ &= E(\beta_0^2 + 2\,\beta_0\,\beta_1\,y_{t-1} + 2\,\beta_0\,\beta_2\,x_{t-1} + 2\,\beta_0\,\epsilon_t + \beta_1^2\,y_{t-1}^2 + \beta_2^2\,x_{t-1}^2 \\ &\quad + 2\,\beta_1\,\beta_2\,y_{t-1}\,x_{t-1} + 2\,\beta_1\,y_{t-1}\,\epsilon_t + 2\,\beta_2\,x_{t-1}\,\epsilon_t + \epsilon_t^2) \\ &= \beta_0^2 + 2\,\beta_0\,\beta_1\,\mu_y + \beta_1^2(\sigma_y^2 + \mu_y^2) + \beta_2^2\,\sigma_x^2 + 2\,\beta_1\,\beta_2\,\sigma_{xy} + \sigma_\epsilon^2 \end{aligned} \tag{B.46}$$

Plugging the obtained solutions into equation B.44, we obtain:

$$\begin{aligned} \sigma_y^2 &= \beta_0^2 + 2\,\beta_0\,\beta_1\,\mu_y + \beta_1^2(\sigma_y^2 + \mu_y^2) + \beta_2^2\,\sigma_x^2 + 2\,\beta_1\,\beta_2\,\sigma_{xy} + \sigma_\epsilon^2 \\ &\quad - (\beta_0^2 + 2\beta_0\,\beta_1\,\mu_y + \beta_1^2\,\mu_y^2), \\ &= \beta_1^2\,\sigma_y^2 + \beta_2^2\,\sigma_x^2 + 2\,\beta_1\,\beta_2\,\sigma_{xy} + \sigma_\epsilon^2 \\ \iff \quad \sigma_y^2 - \beta_1^2\,\sigma_y^2 &= \beta_2^2\,\sigma_x^2 + 2\,\beta_1\,\beta_2\,\sigma_{xy} + \sigma_\epsilon^2 \\ (1-\beta_1^2)\,\sigma_y^2 &= \beta_2^2\,\sigma_x^2 + 2\,\beta_1\,\beta_2\,\sigma_{xy} + \sigma_\epsilon^2 \\ \iff \quad \sigma_y^2 &= \frac{\beta_2^2\,\sigma_x^2 + 2\,\beta_1\,\beta_2\,\sigma_{xy} + \sigma_\epsilon^2}{1-\beta_1^2}. \end{aligned} \tag{B.47}$$

Making use of Mathematica (Wolfram Research & Inc, 2020), we can now combine this expression and the model-implied variance of x_t to express the predictor correlation as:

$$\begin{aligned} \rho_{xy} &= \frac{\sigma_{xy}}{\sqrt{\sigma_x^2}\sqrt{\sigma_y^2}} \\ &= \frac{\sqrt{-\frac{\sigma_\zeta^2}{\gamma^2-1}}\left((\gamma^2-1)\sigma_{\epsilon\zeta} - \beta_2\gamma\sigma_\zeta^2\right)}{(\beta_1\gamma-1)\sigma_\zeta^2\sqrt{\frac{-\beta_1\left(-2\beta_2(\gamma^2-1)\sigma_{\epsilon\zeta} + \beta_2^2\gamma\sigma_\zeta^2 + \gamma(\gamma^2-1)\sigma_\epsilon^2\right) + \beta_2^2\left(-\sigma_\zeta^2\right) + (\gamma^2-1)\sigma_\epsilon^2}{(\beta_1^2-1)(\gamma^2-1)(\beta_1\gamma-1)}}}. \end{aligned} \tag{B.48}$$

Note that the expression for the model-implied predictor covariance presented in Equation B.43 and all following expressions also hold under the model where the covariate exerts a contemporaneous effect, since the sth autocovariance in Equation B.40 is only dependent on s and not on the value t takes. Also, the covariance of the covariate with the preceding innovations derived in Equation B.41 is independent of t, as are the model-implied variances of x_t and y_t.